International Journal of Engineering Research and General Science Volume 3, Issue 3, May-June, 2015
ISSN 2091-2730

Some Notes On Generalized Almost Sasakian Manifolds

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Abstract—In 1960, S. Sasaki [12] discussed on differentiable manifolds which are closely related to almost contact structure. Again in 1961, S. Sasaki and Y. Hatakeyama [13] discussed on differentiable manifolds with certain structures which are closely related to almost contact structure. In 1963, Y. Hatakeyama, Y. Ogawa and S. Tanno [4] studied some properties of manifolds with contact metric structure. In 1963, Y. Hatakeyama [3] discussed on differentiable manifolds with almost contact structures and in 2011, R. Nivas and A. Bajpai [10] studied on generalized Lorentzian Para-Sasakian manifolds. Hayden [5] introduced the idea of metric connection with torsion tensor in a Riemannian manifold. In 1975, Golab [2] studied quarter-symmetric connection in a differentiable manifold. Imai [6] studied the properties of semi-symmetric metric connection in a Riemannian manifold. In 1980, R. S. Mishra and S. N. Pandey [7] discussed on quarter-symmetric metric F-connection. In 1992, Nirmala S. Agashe and Mangala R. Chafle [9] studied semi-symmetric non-metric connection in a Riemannian manifold. In 1982, K. Yano and T. Imai [15] studied on quarter-symmetric metric connections and their curvature tensors. Chaubey S.K. and Pandey A.C. [1] studied the properties of semi-symmetric non-metric connection in Sasakian manifolds. Quarter-symmetric metric connection is also studied by R. N. Singh and S. K. Pandey [14], A. K. Mondal and U. C. De [8] and many others. In this paper, generalized almost Sasakian manifolds have been introduced and some of their properties have been established with generalized almost Co-symplectic manifold. Induced connection in a generalized Sasakian manifold has also been discussed.

Keywords—Generalized almost Sasakian manifold, generalized almost Special Sasakian manifold, generalized almost Co-symplectic manifold and generalized semi-symmetric metric F-connection.

1. INTRODUCTION

Let M_n be an odd $(n = 2m + 1)$ dimensional differentiable manifold, which admits a tensor field $F$of type (1, 1), contravariant vector fields T_i, covariant vector fields A_i, where $i = 3,4,5, \dots (n-1)$ and a metric tensor g, satisfying for arbitrary vector fields $X, Y, Z, \dots$

(1.1) $\quad \overline{\overline{X}} = -X + \sum_{i=3}^{n-1} A_i(X) T_i, \quad \overline{T_i} = 0, \quad A_i(T_i) = 1, \quad \overline{X} \stackrel{\text{def}}{=} FX, \quad A_i(\overline{X}) = 0,$

$\quad$ rank $F = n - i$

(1.2) $\quad g(\overline{X}, \overline{Y}) = g(X, Y) - \sum_{i=3}^{n-1} A_i(X) A_i(Y)$, where $A_i(X) = g(X, T_i)$,

$\quad$ $`F(X,Y) \stackrel{\text{def}}{=} g(\overline{X}, Y) = -`F(Y,X),$

Then M_n is called a generalized almost contact metric manifold (a generalized almost Grayan manifold) and the structure (F, T_i, A_i, g) is called generalized almost contact metric structure [11].

Let D be a Riemannian connection on M_n, then we have [11]

(1.3) (a) $(D_X `F)(\overline{Y}, Z) - (D_X `F)(Y, \overline{Z}) - \sum_{i=3}^{n-1} A_i(Y)(D_X A_i)(Z) - \sum_{i=3}^{n-1} A_i(Z)(D_X A_i)(Y) = 0$

(b) $(D_X `F)(\overline{Y}, \overline{\overline{Z}}) - (D_X `F)(\overline{\overline{Y}}, \overline{Z}) = 0$

(1.4) (a) $(D_X `F)(\overline{Y}, \overline{Z}) + (D_X `F)(Y, Z) - \sum_{i=3}^{n-1} A_i(Y)(D_X A_i)(\overline{Z}) + \sum_{i=3}^{n-1} A_i(Z)(D_X A_i)(\overline{Y}) = 0$

(b) $(D_X `F)(\overline{\overline{Y}}, \overline{\overline{Z}}) + (D_X `F)(\overline{Y}, \overline{Z}) = 0$

A generalized almost contact metric manifold is called a generalized Sasakian manifold, if

(1.5) (a) $\quad i(D_X F)(Y) + \overline{\overline{X}} \sum_{i=3}^{n-1} A_i(Y) \; + g(\overline{X}, \overline{Y}) \sum_{i=3}^{n-1} T_i = 0 \Leftrightarrow$

(b) $\quad i(D_X`F)(Y,Z) - g(\overline{X}, \overline{Z}) \sum_{i=3}^{n-1} A_i(Y) \; + g(\overline{X}, \overline{Y}) \sum_{i=3}^{n-1} A_i(Z) \; = 0 \Leftrightarrow$

(c) $\quad iD_X T_i = \overline{X} + T_i - \sum_{i=3}^{n-1} T_i,$

This gives

(1.6) (a) $\; i(D_X`F)(\overline{Y}, Z) - `F(X,Y) \sum_{i=3}^{n-1} A_i(Z) = 0$

(b) $\; i(D_X`F)\left(\overline{\overline{Y}}, Z\right) - g(\overline{X}, \overline{Y}) \sum_{i=3}^{n-1} A_i(Z) = 0$

(c) $\quad (D_X`F)(Y,Z) - \sum_{i=3}^{n-1} A_i(Y)\,(D_X A_i)(\overline{Z}) + \sum_{i=3}^{n-1} A_i(Z)(D_X A_i)(\overline{Y}) \; = 0$

On this manifold, we have

(1.7) (a) $\; i(D_X A_i)(\overline{Y}) = g(\overline{X}, \; \overline{Y}) \Leftrightarrow$

(b) $\; i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = `F(X,Y)$

A generalized almost contact metric manifold is called a generalized Special Sasakian manifold (a generalized S-Sasakian manifold), if

(1.8) (a) $\quad i(D_X F)(Y) - \overline{X} \sum_{i=3}^{n-1} A_i(Y) \; + `F(X,Y) \sum_{i=3}^{n-1} T_i = 0 \Leftrightarrow$

(b) $\quad i(D_X`F)(Y,Z) - `F(X,Z) \sum_{i=3}^{n-1} A_i(Y) \; + `F(X,Y) \sum_{i=3}^{n-1} A_i(Z) \; = 0 \Leftrightarrow$

(c) $\quad iD_X T_i = \overline{\overline{X}} + T_i - \sum_{i=3}^{n-1} T_i$

This gives

(1.9) (a) $\; i(D_X`F)(\overline{Y}, Z) + g(\overline{X}, \; \overline{Y}) \sum_{i=3}^{n-1} A_i(Z) = 0$

(b) $\; i(D_X`F)\left(\overline{\overline{Y}}, Z\right) - `F(X,Y) \sum_{i=3}^{n-1} A_i(Z) = 0$

(c) $\quad (D_X`F)(Y,Z) - \sum_{i=3}^{n-1} A_i(Y)(D_X A_i)(\overline{Z}) + \sum_{i=3}^{n-1} A_i(Z)(D_X A_i)(\overline{Y}) = 0$

On this manifold, we have

(1.10) (a) $\; i(D_X A_i)(\overline{Y}) = `F(X,Y) \Leftrightarrow$

(b) $\; i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = -g(\overline{X}, \; \overline{Y})$

Nijenhuis tensor in a generalized almost contact metric manifold is given by

(1.11) $\quad `N(X,Y,Z) = (D_{\overline{X}}`F)(Y,Z) - (D_{\overline{Y}}`F)(X,Z) + (D_X`F)(Y, \overline{Z}) - (D_Y`F)(X, \overline{Z})$

Where $\qquad `N(X,Y,Z) \stackrel{\text{def}}{=} g(N(X,Y),Z)$

2. GENERALIZED ALMOST CO-SYMPLECTIC MANIFOLD

A generalized almost contact metric manifold is called a generalized almost Co-symplectic manifold, if

(2.1) $\quad (D_X`F)(Y,Z) + (D_Y`F)(Z,X) + (D_Z`F)(X,Y) + \sum_{i=3}^{n-1} A_i(X)\{(D_Y A_i)(\overline{Z}) - (D_Z A_i)(\overline{Y})\} + \sum_{i=3}^{n-1} A_i(Y)\{(D_Z A_i)(\overline{X}) - (D_X A_i)(\overline{Z})\} + \sum_{i=3}^{n-1} A_i(Z)\{(D_X A_i)(\overline{Y}) - (D_Y A_i)(\overline{X})\} = 0$

3. GENERALIZED ALMOST SASAKIAN MANIFOLD

A generalized almost contact metric manifold is called a generalized almost Sasakian manifold, if

(3.1) $(D_X`F)(Y,Z) + (D_Y`F)(Z,X) + (D_Z`F)(X,Y) = 0$

Therefore, a generalized almost Co-symplectic manifold will be a generalized almost Sasakian manifold, if

(3.2) (a) $i(D_X A_i)(\overline{Y}) = g(\overline{X}, \overline{Y}) \Leftrightarrow$

(b) $i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = `F(X,Y) \Leftrightarrow$

(c) $iD_X T_i = \overline{X} + T_i - \sum_{i=3}^{n-1} T_i$

Barring X, Y, Z in (1.11) and using equations (3.1), (1.3) (b) , we see that a generalized almost Sasakian manifold will be completely integrable, if

(3.3) $(D_{\overline{Z}}`F)\left(\overline{X}, \overline{\overline{Y}}\right) = 0$

4. GENERALIZED ALMOST SPECIAL SASAKIAN MANIFOLD

A generalized almost contact metric manifold is called a generalized almost Special Sasakian manifold (a generalized almost S-Sasakian manifold), if

(4.1) $i(D_X`F)(Y,Z) + i(D_Y`F)(Z,X) + i(D_Z`F)(X,Y)$

$+2\,`F(Y,Z)\sum_{i=3}^{n-1} A_i(X) + 2\,`F(Z,X)\sum_{i=3}^{n-1} A_i(Y) + 2\,`F(X,Y)\sum_{i=3}^{n-1} A_i(Z) = 0$

Therefore, a generalized almost Co-symplectic manifold will be a generalized almost S-Sasakian manifold, if

(4.2) (a) $i(D_X A_i)(\overline{Y}) = `F(X,Y) \Leftrightarrow$

(b) $i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = g(\overline{X}, \overline{Y}) \Leftrightarrow$

(c) $iD_X T_i = \overline{\overline{X}} + T_i - \sum_{i=3}^{n-1} T_i$

Barring X, Y, Z in (1.11) and using equations (4.1), (1.3) (b), we see that a generalized almost S-Sasakian manifold will be completely integrable, if

(4.3) $(D_{\overline{Z}}`F)\left(\overline{\overline{X}}, \overline{Y}\right) = 0$

5. GENERALIZED SEMI-SYMMETRIC METRIC F-CONNECTION IN A GENERALIZED SASAKIAN MANIFOLD

Let M_{2m-1} be submanifold of M_{2m+1} and let $c : M_{2m-1} \to M_{2m+1}$ be the inclusion map such that $d \in M_{2m-1} \to cd \in M_{2m+1}$, where c induces a Jacobian map (linear transformation) $J : T'_{2m-1} \to T'_{2m+1}$.

T'_{2m-1} is tangent space to M_{2m-1} at point d and T'_{2m+1} is tangent space to M_{2m+1} at point cd such that $\hat{X}$ in M_{2m-1} at $d \to J\hat{X}$ in M_{2m+1} at cd

Let $\tilde{g}$ be the induced metric tensor in M_{2m-1} . Then

(5.1) $\tilde{g}(\hat{X}, \hat{Y}) = ((g(J\hat{X}, J\hat{Y}))b$

Let B be an affine connection in a generalized Sasakian manifold M_n, then B is said to be a metric connection if

(5.2) $B_X g = 0$

Therefore, Semi- symmetric metric F-connection B in a generalized Sasakian manifold M_n is given by

(5.3) $iB_X Y = iD_X Y - \sum_{i=3}^{n-1} A_i(Y)FX + \sum_{i=3}^{n-1} g(FX,Y)T_i - 2\sum_{i=3}^{n-1} A_i(X)FY$

Where X and Y are arbitrary vector fields of M_{2m+1}. If

(5.4) $T_i = Jt_i + \rho_i M + \sigma_i N$, where $i = 3,4,5, \ldots\ldots (n-1)$.

Where t_i , $i = 3,4,5, \ldots\ldots (n-1)$, are C^∞ vector fields in M_{2m-1}. M and N are unit normal vectors to M_{2m-1}.

Gauss equation is given by

(5.5) $D_{JX} J\hat{Y} = J(\grave{D}_X \hat{Y}) + p(\hat{X},\hat{Y})M + q(\hat{X},\hat{Y})N$

Where $\grave{D}$ is the connection induced on the submanifold from D and p, q are symmetric bilinear functions in M_{2m-1}.

Similarly

(5.6) $B_{JX} J\hat{Y} = J(\grave{B}_X \hat{Y}) + h(\hat{X},\hat{Y})M + k(\hat{X},\hat{Y})N$,

Where $\grave{B}$ is the connection induced on the submanifold from B and h, k are symmetric bilinear functions in M_{2m-1}.

Inconsequence of (5.3), we have

(5.7) $iB_{JX} J\hat{Y} = iD_{JX} J\hat{Y} - \sum_{i=3}^{n-1} A_i(J\hat{Y})JF\hat{X} + \sum_{i=3}^{n-1} g(JF\hat{X},J\hat{Y})T_i - 2\sum_{i=3}^{n-1} A_i(J\hat{X})JF\hat{Y}$

From (5.5), (5.6) and (5.7), we obtain

(5.8) $iJ(\grave{B}_X \hat{Y}) + ih(\hat{X},\hat{Y})M + ik(\hat{X},\hat{Y})N = iJ(\grave{D}_X \hat{Y}) + ip(\hat{X},\hat{Y})M + iq(\hat{X},\hat{Y})N - \sum_{i=3}^{n-1} A_i(J\hat{Y})JF\hat{X} + \sum_{i=3}^{n-1} g(JF\hat{X},J\hat{Y})T_i - 2\sum_{i=3}^{n-1} A_i(J\hat{X})JF\hat{Y}$

Using (5.4), we get

(5.9) $iJ(\grave{B}_X \hat{Y}) + ih(\hat{X},\hat{Y})M + ik(\hat{X},\hat{Y})N = iJ(\grave{D}_X \hat{Y}) + ip(\hat{X},\hat{Y})M + iq(\hat{X},\hat{Y})N - \sum_{i=3}^{n-1} a_i(\hat{Y})JF\hat{X} + \sum_{i=3}^{n-1}(Jt_i + \rho_i M + \sigma_i N)\, \tilde{g}\,(F\hat{X},\hat{Y}) - 2\sum_{i=3}^{n-1} a_i(\hat{X})JF\hat{Y}$

Where $\tilde{g}(\hat{Y},t_i) \overset{\text{def}}{=} a_i(\hat{Y})$

This gives

(5.10) $i\grave{B}_X \hat{Y} = i\grave{D}_X \hat{Y} - \sum_{i=3}^{n-1} a_i(\hat{Y})F\hat{X} + \sum_{i=3}^{n-1} \tilde{g}(F\hat{X},\hat{Y})t_i - 2\sum_{i=3}^{n-1} a_i(\hat{X})F\hat{Y}$, iff

(5.11) (a) $ih(\hat{X},\hat{Y}) = ip(\hat{X},\hat{Y}) + \sum_{i=3}^{n-1} \rho_i \tilde{g}(F\hat{X},\hat{Y})$

(b) $ik(\hat{X},\hat{Y}) = iq(\hat{X},\hat{Y}) + \sum_{i=3}^{n-1} \sigma_i \tilde{g}(F\hat{X},\hat{Y})$

Therefore, we have

Theorem 5.1 The connection induced on a submanifold of a generalized Sasakian manifold with a generalized Semi-symmetric metric F-connection with respect to unit normal vectors M and N is also generalized Semi- symmetric metric F-connection iff (5.11) holds.

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