Some Notes On Generalized Almost Sasakian Manifolds

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1. INTRODUCTION

Let $M_n$ be an odd $(n = 2m + 1)$ dimensional differentiable manifold, which admits a tensor field $F$ of type $(1, 1)$, contravariant vector fields $T_i$, covariant vector fields $A_i$, where $t = 3, 5, ..., (n - 1)$ and a metric tensor $g$, satisfying for arbitrary vector fields $X, Y, Z, ...$

\[ \nabla X = -X + \sum_{i=3}^{n-1} A_i(X) T_i, \quad T_i = 0, \quad A_i(T_i) = 1, \quad A_i(\nabla X) = 0, \quad \text{rank} \ F = n - i \]

\[ g(\nabla X, Y) = g(X, Y) - \sum_{i=3}^{n-1} A_i(X) A_i(Y), \text{where} \ A_i(X) = g(X, T_i), \]

Then $M_n$ is called a generalized almost contact metric manifold (a generalized almost Grayan manifold) and the structure $(F, T_i, A_i, g)$ is called generalized almost contact metric structure [11].

Let $D$ be a Riemannian connection on $M_n$, then we have [11]

\[ (D_X^*F)(\nabla X, Z) - (D_X^*F)(\nabla Y, Z) = \sum_{i=3}^{n-1} A_i(Y)(D_X A_i)(Z) - \sum_{i=3}^{n-1} A_i(Z)(D_X A_i)(Y) = 0 \]

\[ (b) \quad (D_X^*F)(\nabla X, \nabla Z) = 0 \]

\[ (1.4) \quad (D_X^*F)(\nabla X, Y, Z) = \sum_{i=3}^{n-1} A_i(Y)(D_X A_i)(Z) + \sum_{i=3}^{n-1} A_i(Z)(D_X A_i)(Y) = 0 \]

A generalized almost contact metric manifold is called a generalized Sasakian manifold, if
\( (1.5) \) (a) \( i(D_X F)(Y) + \sum_{i=3}^{n-1} A_i(Y) + g(\bar{X}, \bar{V}) \sum_{i=3}^{n-1} T_i = 0 \Leftrightarrow \)

(b) \( i(D_X F)(Y, Z) = g(\bar{X}, \bar{V}) \sum_{i=3}^{n-1} A_i(Y) + g(\bar{X}, \bar{V}) \sum_{i=3}^{n-1} A_i(Z) = 0 \Leftrightarrow \)

(c) \( iD_X T_i = \bar{X} + T_i - \sum_{i=3}^{n-1} T_i, \)

This gives

\( (1.6) \) (a) \( i(D_X F)(\bar{V}, Z) - \bar{F}(X, Y) \sum_{i=3}^{n-1} A_i(Z) = 0 \)

(b) \( i(D_X F)(\bar{V}, Z) = g(\bar{X}, \bar{V}) \sum_{i=3}^{n-1} A_i(Z) = 0 \)

(c) \( (D_X F)(Y, Z) - \sum_{i=3}^{n-1} A_i(Y) (D_X A_i)(\bar{Z}) + \sum_{i=3}^{n-1} A_i(Z)(D_X A)(\bar{V}) = 0 \)

On this manifold, we have

\( (1.7) \) (a) \( i(D_X A_i)(\bar{V}) = g(\bar{X}, \bar{V}) \Leftrightarrow \)

(b) \( i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = \bar{F}(X, Y) \)

A generalized almost contact metric manifold is called a generalized Special Sasakian manifold (a generalized S-Sasakian manifold), if

\( (1.8) \) (a) \( i(D_X F)(Y) - \bar{X} \sum_{i=3}^{n-1} A_i(Y) + \bar{F}(X, Y) \sum_{i=3}^{n-1} T_i = 0 \Leftrightarrow \)

(b) \( i(D_X F)(Y, Z) - \bar{F}(X, Z) \sum_{i=3}^{n-1} A_i(Y) + \bar{F}(X, Y) \sum_{i=3}^{n-1} A_i(Z) = 0 \Leftrightarrow \)

(c) \( iD_X T_i = \bar{X} + T_i - \sum_{i=3}^{n-1} T_i, \)

This gives

\( (1.9) \) (a) \( i(D_X F)(\bar{V}, Z) + g(\bar{X}, \bar{V}) \sum_{i=3}^{n-1} A_i(Z) = 0 \)

(b) \( i(D_X F)(\bar{V}, Z) - \bar{F}(X, Y) \sum_{i=3}^{n-1} A_i(Z) = 0 \)

(c) \( (D_X F)(Y, Z) - \sum_{i=3}^{n-1} A_i(Y) (D_X A_i)(\bar{Z}) + \sum_{i=3}^{n-1} A_i(Z)(D_X A_i)(\bar{V}) = 0 \)

On this manifold, we have

\( (1.10) \) (a) \( i(D_X A_i)(\bar{V}) = \bar{F}(X, Y) \Leftrightarrow \)

(b) \( i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = -g(\bar{X}, \bar{V}) \)

Nijenhuis tensor in a generalized almost contact metric manifold is given by

\( (1.11) \) \( \bar{N}(X, Y, Z) = (D_X F)(Y, Z) - (D_X F)(X, Z) + (D_X F)(Y, Z) - (D_X F)(X, Z) \)

Where \( \bar{N}(X, Y, Z) \equiv g(N(X, Y), Z) \)

2. GENERALIZED ALMOST CO-SYMPLECTIC MANIFOLD

A generalized almost contact metric manifold is called a generalized almost Co-symplectic manifold, if

\( (2.1) \) \( (D_X F)(Y, Z) + (D_Y F)(Z, X) + (D_Z F)(X, Y) + \sum_{i=3}^{n-1} A_i(X) \{ (D_Y A_i)(\bar{Z}) - (D_Z A_i)(\bar{V}) \} + \sum_{i=3}^{n-1} A_i(Y) \{ (D_Z A_i)(\bar{X}) - (D_X A_i)(\bar{Z}) \} + \sum_{i=3}^{n-1} A_i(Z) \{ (D_X A_i)(\bar{V}) - (D_Y A_i)(\bar{X}) \} = 0 \)
3. GENERALIZED ALMOST SASAKIAN MANIFOLD

A generalized almost contact metric manifold is called a generalized almost Sasakian manifold, if

\[(D_X F)(Y, Z) + (D_Y F)(Z, X) + (D_Z F)(X, Y) = 0\]

Therefore, a generalized almost Co-symplectic manifold will be a generalized almost Sasakian manifold, if

\[i(D_X A_i)(\nabla X) = g(\nabla X, \nabla Y) \Leftrightarrow \]

\[i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = \nabla F(X, Y) \Leftrightarrow \]

\[iD_X T_i = \nabla X + T_i - \sum_{i=3}^{n-1} T_i \]

Barring X, Y, Z in (1.11) and using equations (3.1), (1.3) (b), we see that a generalized almost Sasakian manifold will be completely integrable, if

\[(D_X F)(\nabla X, \nabla Y) = 0\]

4. GENERALIZED ALMOST SPECIAL SASAKIAN MANIFOLD

A generalized almost contact metric manifold is called a generalized almost Special Sasakian manifold (a generalized almost S-Sasakian manifold), if

\[(D_X F)(Y, Z) + i(D_Y F)(Z, X) + i(D_Z F)(X, Y) + 2 \nabla F(Y, Z) \sum_{i=3}^{n-1} A_i(X) + 2 \nabla F(Z, X) \sum_{i=3}^{n-1} A_i(Y) + 2 \nabla F(X, Y) \sum_{i=3}^{n-1} A_i(Z) = 0\]

Therefore, a generalized almost Co-symplectic manifold will be a generalized almost S-Sasakian manifold, if

\[i(D_X A_i)(\nabla X) = \nabla F(X, Y) \Leftrightarrow \]

\[i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = g(\nabla X, \nabla Y) \Leftrightarrow \]

\[iD_X T_i = \nabla X + T_i - \sum_{i=3}^{n-1} T_i \]

Barring X, Y, Z in (1.11) and using equations (4.1), (1.3) (b), we see that a generalized almost S-Sasakian manifold will be completely integrable, if

\[(D_X F)(\nabla X, \nabla Y) = 0\]

5. GENERALIZED SEMI-SYMMETRIC METRIC F-CONNECTION IN A GENERALIZED SASAKIAN MANIFOLD

Let \(M_{2m-1}\) be submanifold of \(M_{2m+1}\) and let \(c : M_{2m-1} \rightarrow M_{2m+1}\) be the inclusion map such that \(d \in M_{2m-1} \rightarrow cd \in M_{2m+1}\), where \(c\) induces a Jacobian map (linear transformation) \(J : T'_{2m-1} \rightarrow T'_{2m+1}\). \(T'_{2m-1}\) is tangent space to \(M_{2m-1}\) at point \(d\) and \(T'_{2m+1}\) is tangent space to \(M_{2m+1}\) at point \(cd\) such that \(\tilde{X}\) in \(M_{2m-1}\) at \(d\) \(\rightarrow f\tilde{X}\) in \(M_{2m+1}\) at \(cd\).

Let \(\tilde{g}\) be the induced metric tensor in \(M_{2m-1}\). Then

\[(5.1) \quad \tilde{g}(\tilde{X}, \tilde{Y}) = ((g(f\tilde{X}, f\tilde{Y}))b)\]

Let \(B\) be an affine connection in a generalized Sasakian manifold \(M_{2m}\), then \(B\) is said to be a metric connection if
Therefore, Semi-symmetric metric F-connection $B$ in a generalized Sasakian manifold $M_n$ is given by

$$iB_X Y = iD_X Y - \sum_{i=3}^{n-1} A_i \left( Y \right) F_X + \sum_{i=3}^{n-1} g \left( F_X, Y \right) T_i - 2 \sum_{i=3}^{n-1} A_i \left( X \right) F_Y$$

Where $X$ and $Y$ are arbitrary vector fields of $M_{2m+1}$. If

$$T_i = Jt_i + \rho_i M + \sigma_i N,$$

Where $t_i$, $i = 3, 4, 5, \ldots, (n-1)$.

Where $t_i$, $i = 3, 4, 5, \ldots, (n-1)$, are $C^\infty$ vector fields in $M_{2m-1}$. $M$ and $N$ are unit normal vectors to $M_{2m-1}$.

Gauss equation is given by

$$D_{JX} \hat{Y} = J(\hat{D}_X \hat{Y}) + p(\hat{X}, \hat{Y}) M + q(\hat{X}, \hat{Y}) N$$

Where $\hat{D}$ is the connection induced on the submanifold from $D$ and $p, q$ are symmetric bilinear functions in $M_{2m-1}$.

Similarly

$$B_{JX} \hat{Y} = J(\hat{B}_X \hat{Y}) + h(\hat{X}, \hat{Y}) M + k(\hat{X}, \hat{Y}) N,$$

Where $\hat{B}$ is the connection induced on the submanifold from $B$ and $h, k$ are symmetric bilinear functions in $M_{2m-1}$.

Inconsequence of (5.3), we have

$$iB_{JX} \hat{Y} = iD_{JX} \hat{Y} - \sum_{i=3}^{n-1} A_i \left( \hat{Y} \right) F_X + \sum_{i=3}^{n-1} g \left( F_X, \hat{Y} \right) T_i - 2 \sum_{i=3}^{n-1} A_i \left( \hat{X} \right) F_Y$$

From (5.5), (5.6) and (5.7), we obtain

$$ij(\hat{B}_X \hat{Y}) + ih(\hat{X}, \hat{Y}) M + ik(\hat{X}, \hat{Y}) N = ij(\hat{D}_X \hat{Y}) + ip(\hat{X}, \hat{Y}) M + iq(\hat{X}, \hat{Y}) N - \sum_{i=3}^{n-1} A_i \left( \hat{Y} \right) F_X + \sum_{i=3}^{n-1} g \left( F_X, \hat{Y} \right) T_i - 2 \sum_{i=3}^{n-1} A_i \left( \hat{X} \right) F_Y$$

Using (5.4), we get

$$ij(\hat{B}_X \hat{Y}) + ih(\hat{X}, \hat{Y}) M + ik(\hat{X}, \hat{Y}) N = ij(\hat{D}_X \hat{Y}) + ip(\hat{X}, \hat{Y}) M + iq(\hat{X}, \hat{Y}) N - \sum_{i=3}^{n-1} A_i \left( \hat{Y} \right) F_X + \sum_{i=3}^{n-1} g \left( F_X, \hat{Y} \right) T_i - 2 \sum_{i=3}^{n-1} A_i \left( \hat{X} \right) F_Y$$

Where $\hat{g}(\hat{Y}, t_i) \equiv a_i(\hat{Y})$

This gives

$$iB_X \hat{Y} = iD_X \hat{Y} - \sum_{i=3}^{n-1} a_i(\hat{Y}) F_X + \sum_{i=3}^{n-1} g \left( F_X, \hat{Y} \right) T_i - 2 \sum_{i=3}^{n-1} a_i(\hat{X}) F_Y, \text{ iff}$$

(5.11) (a) $ih(\hat{X}, \hat{Y}) = ip(\hat{X}, \hat{Y}) + \sum_{i=3}^{n-1} p_i \hat{g}(F_X, \hat{Y})$

(b) $ik(\hat{X}, \hat{Y}) = iq(\hat{X}, \hat{Y}) + \sum_{i=3}^{n-1} a_i \hat{g}(F_X, \hat{Y})$

Therefore, we have

**Theorem 5.1** The connection induced on a submanifold of a generalized Sasakian manifold with a generalized Semi-symmetric metric F-connection with respect to unit normal vectors $M$ and $N$ is also generalized Semi-symmetric metric F-connection iff (5.11) holds.
REFERENCES: