Special Sasakian Manifold with Induced Connection

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Abstract—In 1960, S. Sasaki [7] dicussed on differentiable manifolds which are closely related to almost contact structure. Also in 1961, S. Sasaki and Y. Hatakeyama [8] discussed on differentiable manifolds with certain structures which are closely related to almost contact structure. In 1963, Y. Hatakeyama [1] discussed on differentiable manifolds with almost contact structures and in 2011, R. Nivas and A. Bajpai [5] studied on generalized Lorentzian Para-Sasakian manifolds. Hayden [2] introduced the idea of metric connection with torsion tensor in a Riemannian manifold. In 1980, R. S. Mishra and S. N. Pandey [3] discussed on quarter-symmetric metric F-connection. In 1992, Nirmala S. Agashe and Mangala R. Chafle [4] studied semi-symmetric non-metric connection in a Riemannian manifold. In this paper, generalized nearly Sasakian and generalized nearly special Sasakian manifolds have been introduced and some of their properties have been established with generalized Co-symplectic manifolds. Induced connection in a generalized special Sasakian manifold has also been studied.
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Keywords—Generalized nearly Sasakian manifold, generalized nearly Special Sasakian manifold, generalized Co-symplectic manifolds and generalized semi-symmetric metric F-connection.

1. INTRODUCTION

An n (=2m+1) dimensional differentiable manifold M_n, on which there are defined covariant vector fields A_i,where $i = 3,4,5, \ldots\ldots (n-1)$, the associated contravariant vector fields T_i, where $i = 3,4,5, \ldots\ldots (n-1)$, a tensor field $F$of type (1, 1) and a metric tensor g, satisfying

(1.1) $F^2 = -I_n + \sum_{i=3}^{n-1} A_i \otimes T_i, \quad FT_i = 0, \quad A_i(T_i) = 1, \quad A_i(FX) = 0,$

Rank $F = n - i$

(1.2) $g\,(FX, FY) = g\,(X, Y) - \sum_{i=3}^{n-1} A_i(X)\,A_i(Y)$, where $A_i(X) = g(X, T_i), i = 3,4,5, \ldots\ldots (n-1),$

$`F(X,Y) \stackrel{\text{def}}{=} g(\,FX, Y) = -`F(Y, X),$

Then M_n is called a generalized almost contact metric manifold (a generalized almost Grayan manifold) and the structure $(F,\ T_i,\ A_i,$ $g\ )$ is known as generalized almost contact metric structure [6].

Let D be a Riemannian connection on M_n, then we have [6]

(1.3) (a) $(D_X`F)(FY, Z) - (D_X`F)(Y, FZ) - \sum_{i=3}^{n-1} A_i(Y)(D_X A_i)(Z) - \sum_{i=3}^{n-1} A_i(Z)(D_X A_i)(Y) = 0$

(b) $(D_X`F)(\,FY, F^2 Z) = (D_X`F)(F^2 Y\,, FZ)$

A generalized almost contact metric manifold is called a generalized Sasakian manifold, if

(1.4) (a) $i(D_X F)(Y) + F^2 X \sum_{i=3}^{n-1} A_i(Y) \ + g(\,FX, FY) \sum_{i=3}^{n-1} T_i = 0 \Leftrightarrow$

(b) $i(D_X`F)(Y, Z) - g(\,FX, FZ) \sum_{i=3}^{n-1} A_i(Y) \ + g(\,FX, FY) \sum_{i=3}^{n-1} A_i(Z) \ = 0 \Leftrightarrow$

(c) $iD_X T_i = FX + T_i - \sum_{i=3}^{n-1} T_i,$

From which, we get

(1.5) (a) $i(D_X A_i)(FY) = g(\,FX, FY) \Leftrightarrow$

(b) $i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = `F(X,Y)$

A generalized almost contact metric manifold is called a generalized Special Sasakian manifold (a generalized S-Sasakian manifold), if

(1.6) (a) $i(D_X F)(Y) - FX \sum_{i=3}^{n-1} A_i(Y) + `F(X,Y) \sum_{i=3}^{n-1} T_i = 0 \Leftrightarrow$

(b) $i(D_X `F)(Y,Z) - `F(X,Z) \sum_{i=3}^{n-1} A_i(Y) + `F(X,Y) \sum_{i=3}^{n-1} A_i(Z) = 0 \Leftrightarrow$

(c) $iD_X T_i = F^2 X + T_i - \sum_{i=3}^{n-1} T_i$

From which, we get

(1.7) (a) $i(D_X A_i)(FY) = `F(X,Y) \Leftrightarrow$

(b) $i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = -g(FX, FY)$

A generalized almost contact metric manifold is called a generalized Co-symplectic manifold, if

(1.8) (a) $(D_X F)Y + \sum_{i=3}^{n-1} A_i(Y) F D_X T_i + \sum_{i=3}^{n-1} (D_X A_i)(FY) T_i = 0 \Leftrightarrow$

(b) $(D_X `F)(Y,Z) - \sum_{i=3}^{n-1} A_i(Y)(D_X A_i)(FZ) + \sum_{i=3}^{n-1} A_i(Z)(D_X A_i)(FY) = 0$

Therefore, a generalized Co-symplectic manifold will be a generalized Sasakian manifold, if

(1.9) (a) $i(D_X A_i)(FY) = g(FX, FY) \Leftrightarrow$ (b) $i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = `F(X,Y) \Leftrightarrow$

(c) $iD_X T_i = FX + T_i - \sum_{i=3}^{n-1} T_i$

And a generalized Co-symplectic manifold will be a generalized S-Sasakian manifold, if

(1.10) (a) $i(D_X A_i)(FY) = `F(X,Y) \Leftrightarrow$

(b) $i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = -g(FX, FY) \Leftrightarrow$ (c) $iD_X T_i = F^2 X + T_i - \sum_{i=3}^{n-1} T_i$

Nijenhuis tensor in a generalized almost contact metric manifold is given by

(1.11) $`N(X,Y,Z) = (D_{FX} `F)(Y,Z) - (D_{FY} `F)(X,Z) + (D_X `F)(Y,FZ) - (D_Y `F)(X,FZ)$

Where $`N(X,Y,Z) \stackrel{\text{def}}{=} g(N(X,Y),Z)$

2. GENERALIZED NEARLY SASAKIAN MANIFOLD

A generalized almost contact metric manifold is called a generalized nearly Sasakian manifold, if

(2.1) $i(D_X `F)(Y,Z) - \sum_{i=3}^{n-1} A_i(Y) g(FX, FZ) + \sum_{i=3}^{n-1} A_i(Z) g(FX, FY)$

$= i(D_Y `F)(Z,X) - \sum_{i=3}^{n-1} A_i(Z) g(FX, FY) + \sum_{i=3}^{n-1} A_i(X) g(FY, FZ)$

$= i(D_Z `F)(X,Y) - \sum_{i=3}^{n-1} A_i(X) g(FY, FZ) + \sum_{i=3}^{n-1} A_i(Y) g(FX, FZ)$

From which, we get

(2.2) (a) $i(D_X F)Y + i(D_Y F)X + \sum_{i=3}^{n-1} A_i(Y) F^2 X + \sum_{i=3}^{n-1} A_i(X) F^2 Y + 2 \sum_{i=3}^{n-1} T_i \, g(FX, FY) = 0 \Leftrightarrow$

(b) $i(D_X `F)(Y,Z) + i(D_Y `F)(X,Z) - \sum_{i=3}^{n-1} A_i(Y) g(FX, FZ) - \sum_{i=3}^{n-1} A_i(X) g(FY, FZ) +$

$2 \sum_{i=3}^{n-1} A_i(Z) g(FX, FY) = 0$

From which, we get

(2.3) (a) $i(D_X F)FY + i(D_{FY} F)X - \sum_{i=3}^{n-1} A_i(X) FY - 2 \sum_{i=3}^{n-1} T_i \, `F(X,Y) = 0 \Leftrightarrow$

(b) $i(D_X`F)(FY,Z) - i(D_{FY}`F)(Z,X) - \sum_{i=3}^{n-1} A_i(X)`F(Y,Z) - 2\sum_{i=3}^{n-1} A_i(Z)`F(X,Y) = 0$

(2.4) (a) $i(D_X F)F^2Y + i(D_{F^2Y}F)X - \sum_{i=3}^{n-1} A_i(X)F^2Y - 2\sum_{i=3}^{n-1} T_i\, g(FX,FY) = 0 \Leftrightarrow$

(b) $i(D_X`F)(F^2Y,Z) - i(D_{F^2Y}`F)(Z,X) + \sum_{i=3}^{n-1} A_i(X)g(FY,FZ) - 2\sum_{i=3}^{n-1} A_i(Z)g(FX,FY) = 0$

(2.5) (a) $(D_X F)Y + (D_Y F)X + \sum_{i=3}^{n-1} A_i(Y)\{FD_X T_i - (D_{T_i}F)X\} + \sum_{i=3}^{n-1} A_i(X)\{FD_Y T_i - (D_{T_i}F)Y\} +$

$\sum_{i=3}^{n-1} T_i\{(D_X A_i)(FY) + (D_Y A_i)(FX)\} = 0 \Leftrightarrow$

(b) $(D_X`F)(Y,Z) + (D_Y`F)(X,Z) - \sum_{i=3}^{n-1} A_i(Y)\left\{(D_X A_i)(FZ) - (D_{T_i}`F)(Z,X)\right\} -$

$\sum_{i=3}^{n-1} A_i(X)\left\{(D_Y A_i)(FZ) + (D_{T_i}`F)(Y,Z)\right\} + \sum_{i=3}^{n-1} A_i(Z)\{(D_X A_i)(FY) + (D_Y A_i)(FX)\} = 0$

Pre-multiplying X, Y, Z by F in (1.11) and using equations (2.1), (1.3) (b), we see that a generalized nearly Sasakian manifold will be completely integrable, if

(2.6) $(D_{\overline{X}}`F)\left(\overline{Y}, \overline{\overline{Z}}\right) = (D_{\overline{Y}}`F)(\overline{X}, \overline{\overline{Z}})$

3. GENERALIZED NEARLY SPECIAL SASAKIAN MANIFOLD

A generalized almost contact metric manifold is called a generalized nearly Special Sasakian manifold (a generalized nearly S-Sasakian manifold), if

(3.1) $i(D_X`F)(Y,Z) + \sum_{i=3}^{n-1} A_i(Y)`F(Z,X) + \sum_{i=3}^{n-1} A_i(Z)`F(X,Y)$

$= i(D_Y`F)(Z,X) + \sum_{i=3}^{n-1} A_i(Z)`F(X,Y) + \sum_{i=3}^{n-1} A_i(X)`F(Y,Z)$

$= i(D_Z`F)(X,Y) + \sum_{i=3}^{n-1} A_i(X)`F(Y,Z) + \sum_{i=3}^{n-1} A_i(Y)`F(Z,X)$

From which, we obtain

(3.2) (a) $i(D_X F)Y + i(D_Y F)X - \sum_{i=3}^{n-1} A_i(Y)FX - \sum_{i=3}^{n-1} A_i(X)FY = 0 \Leftrightarrow$

(b) $i(D_X`F)(Y,Z) + i(D_Y`F)(X,Z) + \sum_{i=3}^{n-1} A_i(Y)`F(Z,X) - \sum_{i=3}^{n-1} A_i(X)`F(Y,Z) = 0$

This gives

(3.3) (a) $i(D_X F)FY + i(D_{FY}F)X - \sum_{i=3}^{n-1} A_i(X)F^2Y = 0 \Leftrightarrow$

(b) $i(D_X`F)(FY,Z) - i(D_{FY}`F)(Z,X) + \sum_{i=3}^{n-1} A_i(X)g(FY,FZ) = 0$

(3.4) (a) $i(D_X F)F^2Y + i(D_{F^2Y}F)X + \sum_{i=3}^{n-1} A_i(X)FY = 0 \Leftrightarrow$

(b) $i(D_X`F)(F^2Y,Z) - i(D_{F^2Y}`F)(Z,X) + \sum_{i=3}^{n-1} A_i(X)`F(Y,Z) = 0$

(3.5) (a) $(D_X F)Y + (D_Y F)X + \sum_{i=3}^{n-1} A_i(Y)\{FD_X T_i - (D_{T_i}F)X\} + \sum_{i=3}^{n-1} A_i(X)\{FD_Y T_i - (D_{T_i}F)Y\} = 0 \Leftrightarrow$

(b) $(D_X`F)(Y,Z) + (D_Y`F)(X,Z) - \sum_{i=3}^{n-1} A_i(Y)\left\{(D_X A_i)(FZ) - (D_{T_i}`F)(Z,X)\right\} -$

$\sum_{i=3}^{n-1} A_i(X)\left\{(D_Y A_i)(FZ) + (D_{T_i}`F)(Y,Z)\right\} = 0$

Pre-multiplying X, Y, Z by F in (1.11) and using equations (3.1), (1.3) (b), we see that a generalized nearly S-Sasakian manifold will be completely integrable, if

(3.6) $(D_{\overline{X}}`F)\left(\overline{\overline{Y}}, \overline{Z}\right) = (D_{\overline{Y}}`F)\left(\overline{\overline{X}}, \overline{Z}\right)$

4. GENERALIZED NEARLY CO-SYMPLECTIC MANIFOLD

A generalized almost contact metric manifold is called a generalized nearly Co-symplectic manifold, if

(4.1) $(D_X`F)(Y,Z) - \sum_{i=3}^{n-1} A_i(Y)(D_X A_i)(FZ) + \sum_{i=3}^{n-1} A_i(Z)(D_X A_i)(FY)$

$= (D_Y`F)(Z,X) - \sum_{i=3}^{n-1} A_i(Z)(D_Y A_i)(FX) + \sum_{i=3}^{n-1} A_i(X)(D_Y A_i)(FZ)$

$= (D_Z`F)(X,Y) - \sum_{i=3}^{n-1} A_i(X)(D_Z A_i)(FY) + \sum_{i=3}^{n-1} A_i(Y)(D_Z A_i)(FX)$

Therefore, a generalized nearly Sasakian manifold will be a generalized nearly Co-symplectic manifold, if

(4.2) (a) $i(D_X A_i)(FY) = g(FX, FY) \Leftrightarrow$

(b) $i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = `F(X,Y) \Leftrightarrow$ (c) $iD_X T_i = FX + T_i - \sum_{i=3}^{n-1} T_i$

And a generalized nearly S-Sasakian manifold will be a generalized nearly Co-symplectic manifold, if

(4.3) (a) $i(D_X A_i)(FY) = `F(X,Y) \Leftrightarrow$

(b) $i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = -g(FX, FY) \Leftrightarrow$ (c) $iD_X T_i = F^2X + T_i - \sum_{i=3}^{n-1} T_i$

5. GENERALIZED INDUCED CONNECTION IN A GENERALIZED SPECIAL SASAKIAN MANIFOLD

Let M_{2m-1} be submanifold of M_{2m+1} and let $c : M_{2m-1} \to M_{2m+1}$ be the inclusion map such that

$$d \in M_{2m-1} \to cd \in M_{2m+1} ,$$

Where c induces a Jacobian map (linear transformation) $J : T'_{2m-1} \to T'_{2m+1}$.

T'_{2m-1} is tangent space to M_{2m-1} at point d and T'_{2m+1} is tangent space to M_{2m+1} at point cd such that

$$\hat{X} \text{ in } M_{2m-1} \text{ at } d \to J\hat{X} \text{ in } M_{2m+1} \text{ at } cd$$

Let $\tilde{g}$ be the induced metric tensor in M_{2m-1} , then

(5.1) $\tilde{g}(\hat{X}, \hat{Y}) = ((g(J\hat{X}, J\hat{Y}))b$

Semi- symmetric metric F-connection B in generalized special Sasakian manifold M_n is given by

(5.2) $iB_X Y = iD_X Y + \sum_{i=3}^{n-1} A_i(Y)X - \sum_{i=3}^{n-1} g(X,Y)T_i - 2\sum_{i=3}^{n-1} A_i(X)Y$

Where X and Y are arbitrary vector fields of M_{2m+1}. Let

(5.3) $T_i = Jt_i + \rho_i M + \sigma_i N$, where $i = 3,4,5, \ldots\ldots (n-1)$.

Where t_i , $i = 3,4,5, \ldots\ldots (n-1)$, are C^∞ vector fields in M_{2m-1}. M, N are unit normal vectors to M_{2m-1}.

Denoting by $\grave{D}$ the connection induced on the submanifold from D. Gauss equation is

(5.4) $D_{JX} J\hat{Y} = J(\grave{D}_X \hat{Y}) + p(\hat{X}, \hat{Y})M + q(\hat{X}, \hat{Y})N$

Where p and q are symmetric bilinear functions in M_{2m-1}. Also

(5.5) $B_{JX} J\hat{Y} = J(\grave{B}_X \hat{Y}) + h(\hat{X}, \hat{Y})M + k(\hat{X}, \hat{Y})N$,

Where $\grave{B}$ is the connection induced on the submanifold from B and h, k are symmetric bilinear functions in M_{2m-1}.

Inconsequence of (5.2), we have

(5.6) $iB_{JX}J\hat{Y} = iD_{JX}J\hat{Y} + \sum_{i=3}^{n-1} A_i(J\hat{Y})J\hat{X} - \sum_{i=3}^{n-1} g(J\hat{X},J\hat{Y})T_i - 2\sum_{i=3}^{n-1} A_i(J\hat{X})J\hat{Y}$

Using (5.4), (5.5) and (5.6), we have

(5.7)
$iJ(\grave{B}_X\hat{Y}) + ih(\hat{X},\hat{Y})M + ik(\hat{X},\hat{Y})N =$
$iJ(\grave{D}_X\hat{Y}) + ip(\hat{X},\hat{Y})M + iq(\hat{X},\hat{Y})N + \sum_{i=3}^{n-1} A_i(J\hat{Y})J\hat{X} - \sum_{i=3}^{n-1} g(J\hat{X},J\hat{Y})T_i -$
$2\sum_{i=3}^{n-1} A_i(J\hat{X})J\hat{Y}$

Using (5.3), we get

(5.8) $iJ(\grave{B}_X\hat{Y}) + ih(\hat{X},\hat{Y})M + ik(\hat{X},\hat{Y})N = iJ(\grave{D}_X\hat{Y}) + ip(\hat{X},\hat{Y})M + iq(\hat{X},\hat{Y})N + \sum_{i=3}^{n-1} a_i(\hat{Y})J\hat{X} -$

$$\sum_{i=3}^{n-1}(Jt_i + \rho_i M + \sigma_i N)\,\tilde{g}(\hat{X},\hat{Y}) - 2\sum_{i=3}^{n-1} a_i(\hat{X})J\hat{Y}$$

Where $\tilde{g}(\hat{Y},t_i) \stackrel{\text{def}}{=} a_i(\hat{Y})$

This implies

(5.9) $i\grave{B}_X\hat{Y} = i\grave{D}_X\hat{Y} + \sum_{i=3}^{n-1} a_i(\hat{Y})\hat{X} - \sum_{i=3}^{n-1} \tilde{g}(\hat{X},\hat{Y})t_i - 2\sum_{i=3}^{n-1} a_i(\hat{X})\hat{Y}$

Iff

(5.10) (a) $ih(\hat{X},\hat{Y}) = ip(\hat{X},\hat{Y}) - \sum_{i=3}^{n-1} \rho_i\tilde{g}(\hat{X},\hat{Y})$

(b) $ik(\hat{X},\hat{Y}) = iq(\hat{X},\hat{Y}) - \sum_{i=3}^{n-1} \sigma_i\tilde{g}(\hat{X},\hat{Y})$

Thus, we have

Theorem 5.1 The connection induced on a submanifold of a generalized special Sasakian manifold with a generalized Semi-symmetric metric F-connection with respect to unit normal vectors M and N is also generalized Semi- symmetric metric F-connection iff (5.10) holds.

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