Study On Generalised Structure

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Abstract—In 1989, K. Matsumoto [3] introduced the notion of manifolds with Lorentzian paracontact metric structure similar to the almost paracontact metric structure [8], [9]. Also in 1988, K. Matsumoto and I. Mihai [4] discussed on a certain transformation in a Lorentzian Para-Sasakian manifold and in 2011, R. Nivas and A. Bajpai [7] studied on generalized Lorentzian Para-Sasakian manifolds. T. Suguri and S. Nakayama [11] studied D-conformal deformations on almost contact metric structure. T.Imai [2] discussed on hypersurfaces of a Riemannian manifold with semi-symmetric metric connection. In 1975, Golab [1] studied quarter-symmetric connection in a differentiable manifold. In 1980, R. S. Mishra and S. N. Pandey [5] discussed on quarter-symmetric metric F-connection and in 1982, K. Yano [12] studied on semi symmetric metric connections and their curvature tensors. Symmetric metric connections are also studied by K. Yano and T. Imai [13], Nirmala S. Agashe and Mangala R. Chafle [6], R. N. Singh and S. K. Pandey [10] and many others. The purpose of this paper is to study generalised D-conformal transformation and genaralised induced connection in a generalised Lorentzian contact manifold.

Keywords—Generalised Lorentzian contact manifold, generalised D-conformal transformation, generalised induced connection.

1. INTRODUCTION

An $n(=2m+1)$ dimensional differentiable manifold M_n, on which there are defined a tensor field $F$of type (1, 1), two contravariant vector fields T_1 and T_2 , two covariant vector fields A_1 and A_2 and a Lorentzian metric g, satisfying for arbitrary vector fields $X, Y, Z, \ldots$

(1.1) $\overline{\overline{X}} = -X - A_1(X)T_1 - A_2(X)T_2,\ \overline{T_1} = 0,\ \overline{T_2} = 0,\ A_1(T_1) = 1,\ A_2(T_2) = 1,\ \overline{X} \overset{\text{def}}{=} FX,\ A_1(\overline{X}) = 0,\ A_2(\overline{X}) = 0$, rank $F = n - 2$

(1.2) $g(\overline{X}, \overline{Y}) = g(X, Y) + A_1(X)A_1(Y) + A_2(X)A_2(Y)$, where $A_1(X) = g(X, T_1)$, $A_2(X) = g(X, T_2)$

$`F(X, Y) \overset{\text{def}}{=} g(\overline{X}, Y) = -g(\overline{Y}, X)$,

Then M_n will be called a generalised Lorentzian contact manifold and the structure $(F, T_1, T_2 A_1, A_2, g)$ will be known as Lorentzian contact structure.

It can be easily proved that on a generalised Lorentzian contact manifold, we have

(1.3) (a) $`F(X, Y) + `F(Y, X) = 0$ (b) $`F(\overline{X}, \overline{Y}) = `F(X, Y)$

(1.4) (a) $(D_X `F)(Y, T_1) = -(D_X A_1)(\overline{Y})$ (b) $(D_X `F)(Y, T_2) = -(D_X A_2)(\overline{Y})$

(1.5) (a)
$(D_X `F)(\overline{Y}, Z) - (D_X `F)(Y, \overline{Z}) + A_1(Y)(D_X A_1)(Z) + A_2(Y)(D_X A_2)(Z) + A_1(Z)(D_X A_1)(Y) + A_2(Z)(D_X A_2)(Y) = 0$

(b) $(D_X `F)(\overline{Y}, \overline{\overline{Z}}) = (D_X `F)(\overline{\overline{Y}}, \overline{Z})$

(1.6) (a)
$(D_X `F)(\overline{Y}, \overline{Z}) + (D_X `F)(Y, Z) + A_1(Y)(D_X A_1)(\overline{Z}) + A_2(Y)(D_X A_2)(\overline{Z}) - A_1(Z)(D_X A_1)(\overline{Y}) - A_2(Z)(D_X A_2)(\overline{Y}) = 0$

(b) $(D_X`F)\left(\overline{\overline{Y}}, \overline{\overline{Z}}\right) + (D_X`F)(\overline{Y}, \overline{Z}) = 0$

Where D is the Riemannian connection on M_n.

2. GENERALIZED D- CONFORMAL TRANSFORMATION

Let the corresponding Jacobian map J of the transformation b transforms the structure $(F, T_1, T_2, A_1, A_2, g)$ to the structure $(F, V_1, V_2, v_1, v_2, h)$ such that

(2.1) (a) $J\overline{Z} = \overline{JZ}$ (b) $h(JX, JY)ob = e^{\sigma}\, g\left(\overline{X}, \overline{Y}\right) - e^{2\sigma} A_1(X)A_1(Y) - e^{2\sigma} A_2(X)A_2(Y)$

(c) $V_1 = e^{-\sigma} JT_1,\ V_2 = e^{-\sigma} JT_2$ (d) $v_1(JX)ob = e^{\sigma} A_1(X),\ v_2(JX)ob = e^{\sigma} A_2(X)$

Where σ is a differentiable function on M_n, then the transformation is said to be generalised D-conformal transformation. If σ is a constant, the transformation is known as D-homothetic.

Theorem 2.1 The structure $(F, V_1, V_2, v_1, v_2, h)$ is generalised Lorentzian contact.

Proof. Inconsequence of (1.1), (1.2), (2.1) (b) and (2.1) (d), we have

$$h(J\overline{X}, J\overline{Y})ob = e^{\sigma}\, g\left(\overline{X}, \overline{Y}\right) = h(JX, JY)ob + e^{2\sigma} A_1(X)A_1(Y) + e^{2\sigma} A_2(X)A_2(Y)$$

$$= h(JX, JY)ob + \{v_1(JX)ob\}\{v_1(JY)ob\} + \{v_2(JX)ob\}\{v_2(JY)ob\}$$

This implies

(2.2) $h(J\overline{X}, J\overline{Y}) = h(JX, JY) + v_1(JX)\, v_1(JY) + v_2(JX)\, v_2(JY)$

Making the use of (1.1), (2.1) (a), (2.1) (c) and (2.1) (d), we get

(2.3) $\overline{\overline{JX}} = J\overline{\overline{X}} = -JX - A_1(X)JT_1 - A_2(X)JT_2 = -JX - \{v_1(JX)ob\}V_1 - \{v_2(JX)ob\}V_2$

Also

(2.4) $\overline{V_1} = e^{-\sigma}\overline{JT_1} = 0,\ \overline{V_2} = e^{-\sigma}\overline{JT_2} = 0$

Equations (2.2), (2.3) and (2.4) prove the statement.

Theorem 2.2 Let E and D be the Riemannian connections with respect to h and g such that

(2.5) (a) $E_{JX}JY = JD_XY + JH(X, Y)$ (b) $`H(X, Y, Z) \overset{\text{def}}{=} g(H(X, Y), Z)$

Then

(2.6) $2E_{JX}JY =$

$2JD_XY - J[2e^{\sigma}\{(X\sigma)\, A_1(Y)\, T_1 + (X\sigma)\, A_2(Y)\, T_2 + (Y\sigma)\, A_1(X)\, T_1 + (Y\sigma)\, A_2(X)\, T_2 - ({}^{-1}G\nabla\sigma)\, A_1(X)\, A_1(Y) - ({}^{-1}G\nabla\sigma)\, A_2(X)\, A_2(Y)\} + (e^{\sigma} - 1)\left\{(D_XA_1)(Y) + (D_YA_1)(X) - 2A_1\left(H(X, Y)\right)\right\}T_1 + (e^{\sigma} - 1)\left\{(D_XA_2)(Y) + (D_YA_2)(X) - 2A_2\left(H(X, Y)\right)\right\}T_2 + (e^{\sigma} - 1)\{A_1(X)(D_YT_1) + A_2(X)(D_YT_2) + A_1(Y)(D_XT_1) + A_2(Y)(D_XT_2) - A_1(X)({}^{-1}G\nabla A_1)(Y) - A_2(X)({}^{-1}G\nabla A_2)(Y) - A_1(Y)({}^{-1}G\nabla A_1)(X) - A_2(Y)({}^{-1}G\nabla A_2)(X)\}]$

Proof. Inconsequence of (2.1) (b), we have

$$JX\left(h(JY, JZ)\right)ob = X\left\{e^{\sigma}\, g\left(\overline{Y}, \overline{Z}\right) - e^{2\sigma} A_1(Y)A_1(Z) - e^{2\sigma} A_2(Y)A_2(Z)\right\}$$

From (2.1) (b) and (2.5), we have

(2.7) $h\left(E_{JX}JY, JZ\right)ob + h\left(JY, E_{JX}JZ\right)ob = e^{\sigma}\, g\left(\overline{D_XY}, \overline{Z}\right) - e^{2\sigma} A_1(D_XY)A_1(Z) - e^{2\sigma} A_2(D_XY)A_2(Z) + e^{\sigma}\, g\left(\overline{H(X, Y)}, \overline{Z}\right) - e^{2\sigma} A_1\left(H(X, Y)\right)A_1(Z) - e^{2\sigma} A_2\left(H(X, Y)\right)A_2(Z) + e^{\sigma}\, g\left(\overline{Y}, \overline{H(X, Z)}\right) - e^{2\sigma} A_1(Y)A_1\left(H(X, Z)\right) - e^{2\sigma} A_2(Y)A_2\left(H(X, Z)\right) + e^{\sigma}\, g\left(\overline{Y}, \overline{D_XZ}\right) - e^{2\sigma} A_1(D_XZ)A_1(Y) - e^{2\sigma} A_2(D_XZ)A_2(Y)$

Also

(2.8) $\quad h\left(E_{JX}JY, JZ\right)ob + h\left(JY, E_{JX}JZ\right)ob =$

$$(X\sigma)e^{\sigma}\,\mathrm{g}\left(\overline{Y},\overline{Z}\right) + e^{\sigma}\,\mathrm{g}\left(D_X\overline{Y},\overline{Z}\right) + e^{\sigma}\,\mathrm{g}\left(\overline{Y},D_X\overline{Z}\right) - 2(X\sigma)e^{2\sigma}A_1(Y)A_1(Z) - e^{2\sigma}(D_XA_1)(Y)A_1(Z) - e^{2\sigma}(D_XA_1)(Z)A_1(Y) - e^{2\sigma}A_1(D_XY)A_1(Z) - e^{2\sigma}A_1(D_XZ)A_1(Y) - 2(X\sigma)e^{2\sigma}A_2(Y)A_2(Z) - e^{2\sigma}(D_XA_2)(Y)A_2(Z) - e^{2\sigma}(D_XA_2)(Z)A_2(Y) - e^{2\sigma}A_2(D_XY)A_2(Z) - e^{2\sigma}A_2(D_XZ)A_2(Y)$$

Equations (1.5) (a), (2.7) and (2.8) imply

(2.9)

$$(X\sigma)\mathrm{g}\left(\overline{Y},\overline{Z}\right) - 2(X\sigma)e^{\sigma}A_1(Y)A_1(Z) - 2(X\sigma)e^{\sigma}A_2(Y)A_2(Z) - (e^{\sigma}-1)\{(D_XA_1)(Y)A_1(Z) + (D_XA_2)(Y)A_2(Z) + (D_XA_1)(Z)A_1(Y) + (D_XA_2)(Z)A_2(Y)\} = `H(X,Y,Z) + `H(X,Z,Y) - (e^{\sigma}-1)\{A_1\left(H(X,Y)\right)A_1(Z) + A_2\left(H(X,Y)\right)A_2(Z) + A_1\left(H(X,Z)\right)A_1(Y) + A_2\left(H(X,Z)\right)A_2(Y)\}$$

Writing two other equations by cyclic permutation of X, Y, Z and subtracting the third equation from the sum of the first two equations and using symmetry of $`H$ in the first two slots, we get

(2.10)

$$2`H(X,Y,Z) = -2e^{\sigma}\{(X\sigma)A_1(Y)A_1(Z) + (X\sigma)A_2(Y)A_2(Z) + (Y\sigma)A_1(Z)A_1(X) + (Y\sigma)A_2(Z)A_2(X) - (Z\sigma)A_1(X)A_1(Y) - (Z\sigma)A_2(X)A_2(Y)\} - (e^{\sigma}-1)\left[A_1(Z)\{(D_XA_1)(Y) + (D_YA_1)(X) - 2A_1\left(H(X,Y)\right)\} + A_2(Z)\{(D_XA_2)(Y) + (D_YA_2)(X) - 2A_2\left(H(X,Y)\right)\} + A_1(X)\{(D_YA_1)(Z) - (D_ZA_1)(Y)\} + A_2(X)\{(D_YA_2)(Z) - (D_ZA_2)(Y)\} + A_1(Y)\{(D_XA_1)(Z) - (D_ZA_1)(X)\} + A_2(Y)\{(D_XA_2)(Z) - (D_ZA_2)(X)\}\right]$$

This gives

(2.11)

$$2H(X,Y) = -2e^{\sigma}\left[(X\sigma)A_1(Y)T_1 + (X\sigma)A_2(Y)\,T_2 + (Y\sigma)A_1(X)T_1 + (Y\sigma)A_2(X)\,T_2 - ({}^{-1}G\nabla\sigma)A_1(X)A_1(Y) - ({}^{-1}G\nabla\sigma)A_2(X)A_2(Y)\right] - (e^{\sigma}-1)\left[\{(D_XA_1)(Y) + (D_YA_1)(X) - 2A_1\left(H(X,Y)\right)\}T_1 + \{(D_XA_2)(Y) + (D_YA_2)(X) - 2A_2\left(H(X,Y)\right)\}\,T_2 + A_1(X)(D_YT_1) + A_2(X)(D_Y\,T_2) + A_1(Y)(D_XT_1) + A_2(Y)(D_X\,T_2) - A_1(X)({}^{-1}G\nabla A_1)(Y) - A_2(X)({}^{-1}G\nabla A_2)(Y) - A_1(Y)({}^{-1}G\nabla A_1)(X) - A_2(Y)({}^{-1}G\nabla A_2)(X)\right]$$

Substitution of (2.11) into (2.5) (a) gives (2.6).

3. GENERALIZED INDUCED CONNECTION

Let M_{2m-1} be submanifold of M_{2m+1} and let $c : M_{2m-1} \rightarrow M_{2m+1}$ be the inclusion map such that

$$d \in M_{2m-1} \rightarrow cd \in M_{2m+1}\ ,$$

Where c induces a linear transformation (Jacobian map) $\quad J : T'_{2m-1} \rightarrow T'_{2m+1}$.

T'_{2m-1} is a tangent space to M_{2m-1} at point d and T'_{2m+1} is a tangent space to M_{2m+1} at point cd such that

$$\hat{X} \text{ in } M_{2m-1} \text{ at } d \rightarrow J\hat{X} \text{ in } M_{2m+1} \text{ at } cd$$

Let $\tilde{g}$ be the induced tensor field in M_{2m-1} . Then we have

(3.1) $\quad \tilde{g}\left(\hat{X},\hat{Y}\right) = ((\,g(\,J\hat{X},J\hat{Y}\,))b$

A linear connection B in a generalised Lorentzian contact manifold is said to be a generalised Ricci quarter symmetric metric connection, if

(3.2) (a) $(B_X g)(Y,Z) = 0$ and

(b) $S(X,Y) = B_X Y - B_Y X - [X,Y] = A_1(Y)LX + A_2(Y)LX - A_1(X)LY - A_2(X)LY,$

Where $S(X,Y)$ is a torsion tensor of B and L is the (1, 1) Ricci tensor defined by

(3.3) $g(LX,Y) = Ric(X,Y)$

Then generalised Ricci quarter symmetric metric connection B is given by

(3.4) $2B_X Y = 2D_X Y + A_1(Y)LX + A_2(Y)LX - Ric(X,Y)T_1 - Ric(X,Y)T_2$,

Where X and Y are arbitrary vector fields of M_{2m+1}. If

(3.5) (a) $T_1 = Jt_1 + \rho_1 M + \sigma_1 N$ and

(b) $T_2 = Jt_2 + \rho_2 M + \sigma_2 N$

Where t_1 and t_2 are C^∞ vector fields in M_{2m-1} and M and N are unit normal vectors to M_{2m-1}.

Denoting by $\grave{D}$ the connection induced on the submanifold from D, we have Gauss equation

(3.6) $D_{JX} J\hat{Y} = J(\grave{D}_X \hat{Y}) + h(\hat{X},\hat{Y})M + k(\hat{X},\hat{Y})N$

Where h and k are symmetric bilinear functions in M_{2m-1}. Similarly we have

(3.7) $B_{JX} J\hat{Y} = J(\grave{B}_X \hat{Y}) + m(\hat{X},\hat{Y})M + n(\hat{X},\hat{Y})N$,

Where $\grave{B}$ is the connection induced on the submanifold from B and m and n are symmetric bilinear functions in M_{2m-1}

Inconsequence of (3.4), we have

(3.8) $2B_{JX} J\hat{Y} = 2D_{JX} J\hat{Y} + A_1(J\hat{Y})JL\hat{X} + A_2(J\hat{Y})JL\hat{X} - Ric(J\hat{X},J\hat{Y})T_1 - Ric(J\hat{X},J\hat{Y})T_2$

Using (3.6), (3.7) and (3.8), we get

(3.9) $2J(\grave{B}_X \hat{Y}) + 2m(\hat{X},\hat{Y})M + 2n(\hat{X},\hat{Y})N = 2J(\grave{D}_X \hat{Y}) + 2h(\hat{X},\hat{Y})M + 2k(\hat{X},\hat{Y})N + A_1(J\hat{Y})JL\hat{X} + A_2(J\hat{Y})JL\hat{X} -$

$-Ric(J\hat{X},J\hat{Y})T_1 - Ric(J\hat{X},J\hat{Y})T_2$

Using (3.5) (a) and (3.5) (b), we obtain

(3.10) $2J(\grave{B}_X \hat{Y}) + 2m(\hat{X},\hat{Y})M + 2n(\hat{X},\hat{Y})N = 2J(\grave{D}_X \hat{Y}) + 2h(\hat{X},\hat{Y})M + 2k(\hat{X},\hat{Y})N + a_1(\hat{Y})JL\hat{X} + a_2(\hat{Y})JL\hat{X} -$

$(Jt_1 + \rho_1 M + \sigma_1 N)\tilde{R}ic(\hat{X},\hat{Y}) - (Jt_2 + \rho_2 M + \sigma_2 N)\tilde{R}ic(\hat{X},\hat{Y})$

Where $\tilde{g}(\hat{Y},t_1) \stackrel{\text{def}}{=} a_1(\hat{Y})$ and $\tilde{g}(\hat{Y},t_2) \stackrel{\text{def}}{=} a_2(\hat{Y})$

This gives

(3.11) $2\grave{B}_X \hat{Y} = 2\grave{D}_X \hat{Y} + a_1(\hat{Y})L\hat{X} + a_2(\hat{Y})L\hat{X} - \tilde{R}ic(\hat{X},\hat{Y})t_1 - \tilde{R}ic(\hat{X},\hat{Y})t_2$

Iff

(3.12) (a) $2m(\hat{X},\hat{Y}) = 2h(\hat{X},\hat{Y}) - \rho_1 \tilde{R}ic(\hat{X},\hat{Y}) - \rho_2 \tilde{R}ic(\hat{X},\hat{Y})$

(b) $2n(\hat{X},\hat{Y}) = 2k(\hat{X},\hat{Y}) - \sigma_1 \tilde{R}ic(\hat{X},\hat{Y}) - \sigma_2 \tilde{R}ic(\hat{X},\hat{Y})$

Thus we have

Theorem 3.1 The connection induced on a submanifold of a generalised Lorentzian contact manifold with a generalised Ricci quarter symmetric metric connection with respect to unit normal vectors M and N is also Ricci quarter symmetric iff (3.12) holds.

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