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SOME CHARACTERIZATIONS OF QUADRATIC HAZARD RATE – GEOMETRIC (QHR-G) DISTRIBUTION

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ABSTRACT

A more flexible quadratic hazard rate-geometric (QHR-G) distribution having four parameters is characterized through the hazard function, Mills ratio, the reverse hazard function, Elasticity function and ratio of truncated moments. The applications of characterizations of QHR-G distribution will be constructive for scientists in diverse areas of science.

Keywords: Quadratic Hazard Rate; Mills Ratio; Elasticity; Geometric Distribution; Characterization,

1. INTRODUCTION

Bain (1974) developed quadratic hazard rate (QHR) distribution from the following quadratic function

$$A(x)=\alpha+\beta x+\gamma x^2, x>0. \quad (1)$$

The cumulative distribution function (cdf) of random variable X with QHR distribution and parameters α, β *and* γ is

$$G(x)=1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}, \alpha>0,\ \gamma>0,\ \beta>-2\sqrt{\alpha\gamma},\ x\geq 0. \quad (2)$$

The probability density function (pdf) of random variable X with QHR distribution and parameters α, β *and* γ is

$$g(x)=\left(\alpha+\beta x+\gamma x^2\right)e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}. \quad (3)$$

The geometric distribution with parameter θ has the following probability mass function

$$P(N=n)=(1-\theta)\theta^{n-1},\ 0\leq\theta<1 \quad n=1,2,... \quad (4)$$

2. QUADRATIC HAZARD RATE GEOMETRIC DISTRIBUTION

The quadratic hazard rate-geometric (QHR-G) distribution is developed with mixture of QHR distribution and geometric distribution. The cdf for mixture of continuous probability distribution and geometric distribution is given as

$$F_X(x)=(1-\theta)G(x)\left(1-\theta G(x)\right)^{-1}. \quad (5)$$

The pdf for mixture of continuous probability distribution and geometric distribution is given as

$$f_X(x)=(1-\theta)g(x)\left(1-\theta G(x)\right)^{-2}. \quad (6)$$

Okasha et.el (2016) studied QHR-G distribution along with its applications in reliability. But characterizations of QHR-G distribution are yet to do.

2.1 Probability Density Function of QHR-G Distribution

The pdf of random variable X with QHR-G distribution and parameters α, β, γ *and* $\theta\in(0,1)$ is

$$f_X(x)=(1-\theta)\left(\alpha+\beta x+\gamma x^2\right)e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\left(1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)^{-2}, x>0. \quad (7)$$

2.2 Cumulative Distribution Function and Other Properties of QHR-G Distribution

The cdf for random variable X with QHR-G distribution and parameters α,β,γ *and* $\theta\in(0,1)$ is

$$F_X(x)=(1-\theta)\left[1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right]\left[1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right]^{-1}. \quad (8)$$

The hazard rate function for random variable X with QHR-G distribution is

$$h_F(x)=\frac{(1-\theta)\left(\alpha+\beta x+\gamma x^2\right)}{1-\theta-\theta e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}}. \quad (9)$$

The reverse hazard rate function for random variable X with QHR-G distribution is

$$r_F(x)=\left(\alpha+\beta x+\gamma x^2\right)\left(1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)^{-1}. \quad (10)$$

The mills ratio of QHR-G distribution is

$$m(x)=(1-\theta)\left(\alpha+\beta x+\gamma x^2\right)\left(1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)^{-1}. \quad (11)$$

The elasticity of QHR-G distribution is given by

$$e(x)=\frac{d\ln F(x)}{d\ln x}=x\,r_F(x)=x\left(\alpha+\beta x+\gamma x^2\right)\left(1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)^{-1}. \quad (12)$$

The elasticity of QHR-G distribution shows behavior of accumulation of probability in the domain of random variable.

The rest of paper is composed as follows. QHR-G distribution is characterized through the hazard function, Mills ratio, the reverse hazard function, Elasticity function and ratio of truncated moments.

3. CHARACTERIZATION

In order to develop a stochastic function for a certain problem, it is necessary to know whether function fulfills the theory of specific underlying probability distribution, it is required to study characterizations of specific probability distribution. Different characterization techniques have developed. Glanzel (1987, 1988 and 1990), Hamedani (1993, 2002, 2011 and 2015), Ahsanullah and Hamedani (2007, 2012), Ahsanullah et al. (2013), Shakil et al. (2014), and Merovci et al. (2016) have worked on characterization.

3.1 Characterization Based On Hazard Function

Definition 3.1.1: Let $X:\Omega\to(0,\infty)$ be a continuous random variable with pdf $f(x)$ if and only if the hazard function $h_F(x)$, of a twice differentiable function **F**, satisfies equation

$$\frac{d}{dx}\left[\ln f(x)\right]=\frac{h_F'(x)}{h_F(x)}-h_F(x).$$

Proposition 3.1.1

Let $X:\Omega\to(0,\infty)$ be a continuous random variable with pdf (7) if and only if the hazard function (9) twice differentiable function satisfies equation

$$\left[h_F'(x)\left(1-\theta+\theta e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)-h_F(x)\left(\theta\left(\alpha+\beta x+\gamma x^2\right)e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right]=\left[\beta+2\gamma x-\theta\left(\beta+2\gamma x\right)\right]$$

Proof

For random variable X having QHRG distribution with hazard rate function (9), we obtain the following equation

$$\left[h_X'(x)\left(1-\theta+\theta e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)-h_X(x)\left(\theta\left(\alpha+\beta x+\gamma x^2\right)e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right]=\left[\beta+2\gamma x-\theta\left(\beta+2\gamma x\right)\right].$$

After simplification we obtain as

$$\frac{d}{dx}\left[h_X(x)\left(1-\theta+\theta e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right]=\frac{d}{dx}\left[(1-\theta)\left(\alpha+\beta x+\gamma x^2\right)\right].$$

From above equation we obtain as $$h_X(x)=\frac{(1-\theta)\left(\alpha+\beta x+\gamma x^2\right)}{1-\theta-\theta e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}}. \quad (13)$$

After manipulation, integrating (13), and simplifying, we obtain as

$$F_X(x)=(1-\theta)\left[1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right]\left[1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right]^{-1}.$$

This is cdf of QHRG distribution.

3.2 Characterization Based on Mills Ratio

In this sub-section, we characterize QHR-G distribution via the Mills ratio.

Definition 3.2.1: Let $X:\Omega\to(0,\infty)$ be a continuous random variable having absolutely continuous cdf $F(x)$ and pdf $f(x)$ if and only if the Mills ratio, $m(x)$, of a twice differentiable function **F**, satisfies equation

$$\frac{d\left[\ln f(x)\right]}{dx}+\frac{\left[m'(x)+1\right]}{m(x)}=0.$$

Proposition 3.2.1: Let $X:\Omega\to(0,\infty)$ be continuous random variable .The pdf of X is (7) if and only if the mills ratio fulfills the first order differential equation

$$m'(x)\left(1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)-m(x)\theta\left(\alpha+\beta x+\gamma x^2\right)e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}=(1-\theta)(\beta+2\gamma x).$$

Proof

If X has pdf (7), then above differential equation surely holds. Now if differential equation holds

then $$\frac{d}{dx}\left[m(x)\left(1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)\right]=(1-\theta)\frac{d}{dx}\left[\left(\alpha+\beta x+\gamma x^2\right)\right].$$

After integration of above equation, we reach at

$$m(x)\left(1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)=(1-\theta)\left(\alpha+\beta x+\gamma x^2\right),$$

$$m(x)=(1-\theta)\left(\alpha+\beta x+\gamma x^2\right)\left(1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)^{-1}.$$

This is the Mills ratio of QHR-G distribution.

Proposition 3.2.2: Let $X:\Omega\to(0,\infty)$ be continuous random variable .The pdf of X is (7) if and only if the mills ratio fulfills the first order differential equation

$$m'(x)\left(\alpha+\beta x+\gamma x^2\right)^{-1}-\left(\beta+2\gamma x\right)\left(\alpha+\beta x+\gamma x^2\right)^{-2}=(1-\theta)\theta\left(\alpha+\beta x+\gamma x^2\right)e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\left(1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)^{-2}.$$

Proof

If X has pdf (7), then above differential equation surely holds. Now if differential equation holds

then
$$\frac{d}{dx}\left[m(x)\left(\alpha+\beta x+\gamma x^2\right)^{-1}\right]=(1-\theta)\frac{d}{dx}\left(1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)^{-1},$$

or
$$m(x)\left(\alpha+\beta x+\gamma x^2\right)^{-1}=(1-\theta)\left(1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)^{-1}.$$

$$m(x)=(1-\theta)\left(\alpha+\beta x+\gamma x^2\right)\left(1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)^{-1}$$
, this is Mills ratio of QHR-G distribution.

3.3 Characterization Based Reverse Hazard Function

In this sub-section, we characterize QHR-G distribution via reverse hazard function.

Definition 3.3.1: Let $X:\Omega\to(0,\infty)$ be a continuous random variable with pdf $f(x)$ if and only if the reverse hazard function $r_F(x)$, of a twice differentiable function **F**, satisfies equation

$$\frac{d}{dx}\left[\ln f(x)\right]=\frac{r_F^{/}(x)}{r_F(x)}+r_F(x).$$

Proposition 3.3.1: Let $X:\Omega\to(0,\infty)$ be a continuous random variable with pdf (7) if and only if the reverse hazard function (10) twice differentiable function satisfies equation

$$\left[r_F^{/}(x)\left(1-\theta+\theta e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)+r_F(x)\theta\left(\alpha+\beta x+\gamma x^2\right)^2e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right]=\left[\beta+2\gamma x\right].$$

Proof

For random variable X having QHR-G distribution with reverse hazard rate function (10), we obtain the following equation

$$\left[r_F^{/}(x)\left(1-\theta+\theta e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)+r_F(x)\theta\left(\alpha+\beta x+\gamma x^2\right)^2e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right]=\left[\beta+2\gamma x\right].$$

After simplification we obtain as
$$\frac{d}{dx}\left[r_F(x)\left(1-\theta+\theta e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right]=\frac{d}{dx}\left[\alpha+\beta x+\gamma x^2\right].$$

From above equation we obtain as $r_F(x)=\frac{f_X(x)}{F_X(x)}=\frac{\left(\alpha+\beta x+\gamma x^2\right)}{\left(1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)}$.

After manipulation, integrating above equation, and simplifying, we obtain as

$$F_X(x)=(1-\theta)\left[1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right]\left[1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right]^{-1}.$$

This is cdf of QHRG distribution.

3.4 Characterization Based On Elasticity Function

In this sub-section, we characterize QHR-G distribution via elasticity function.

Definition 3.4.1: Let $X:\Omega\to(0,\infty)$ be a continuous random variable having absolutely continuous $F(x)$ and pdf $f(x)$ provided the elasticity function $e_F(x)$ is twice differentiable function satisfying differential equation

$$\frac{d}{dx}\left[\ln f(x)\right]=\frac{e^{/}(x)}{e(x)}+\frac{e(x)}{x}-\frac{1}{x}.$$

Proposition 3.4.1 Let $X:\Omega\to(0,\infty)$ be continuous random variable. The pdf of X is (6) provided that its elasticity function, $e_F(x)$ satisfies the first order differential equation

$$e'(x)\left(1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)+e(x)\theta\left(\alpha+\beta x+\gamma x^2\right)e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}=\left(\alpha+2\beta x+3\gamma x^2\right). \quad (14)$$

Proof

If X has pdf (7), then (14) surely holds. Now if (14) holds, then

$$\frac{d}{dx}\left[e(x)\left(1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)\right]=\frac{d}{dx}\left[\left(\alpha x+\beta x^2+\gamma x^3\right)\right].$$

$$e(x)=x\left(\alpha+\beta x+\gamma x^2\right)\left(1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)^{-1},$$

which is the elasticity function of QHR-G distribution.

Proposition 3.4.2: Let $X:\Omega\to(0,\infty)$ be continuous random variable. The pdf of X is (6) provided that its elasticity function, $e_F(x)$ satisfies the first order differential equation

$$e'(x)\left(\alpha+\beta x+\gamma x^2\right)^{-1}-\left(\beta+2\gamma x\right)\left(\alpha+\beta x+\gamma x^2\right)^{-2}e(x)=\left(1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)^{-1}+x\theta\left(\alpha+\beta x+\gamma x^2\right)e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\left(1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)^{-2}. \quad (15)$$

Proof

If X has pdf (7), then (15) surely holds. Now if (15) holds, then

$$\frac{d}{dx}\left[e(x)\left(\alpha+\beta x+\gamma x^2\right)^{-1}\right]=\frac{d}{dx}\left[x\left(1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)^{-1}\right],$$

or $e(x)=x\left(\alpha+\beta x+\gamma x^2\right)\left(1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)^{-1}$, the elasticity function of QHR-G distribution.

3.5 Characterization through Ratio of Truncated Moments

In this section, we characterize GLBXII distribution using Theorem 1 (Glanzel; 1987) on the basis of simple relationship between two functions of X. Theorem 1 is given in appendix A.

Proposition 3.5.1: Suppose that random variable $X:\Omega\to(0,\infty)$ is continuous. Let

$h_1(x)=\frac{1}{(1-\theta)}\left(1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)^2$ and $h_2(x)=\frac{2e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}}{(1-\theta)\left(1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)^{-2}}$, $x>0.$

The pdf of X is (7) if and only if $p(x)$ has the form $p(x)=\exp\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)\ x>0$.

Proof
For random variable X having QHRG distribution with pdf (7) and cdf (8), we proceeds as

$$\left(1-F(x)\right)E\left(h_1(x)/X\geq x\right)=\mathrm{e}^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\quad x>0$$

$$\left(1-F(x)\right)E\left(h_2(x)/X\geq x\right)=\mathrm{e}^{-2\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\quad x>0$$

$p(x)=e^{\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}$ and $p'(x)=\left(\alpha+\beta x+\gamma x^2\right)\exp\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)$

$s'(t)=\frac{p'(t)h_2(t)}{p(t)h_2(t)-h_1(t)}=2\left(\alpha+\beta x+\gamma x^2\right)$ and $s(t)=2\left(\alpha x+\beta\frac{x^2}{2}+\gamma\frac{x^3}{3}\right)$

Therefore in the light of Theorem 1, X has pdf (7)

Corollary3.5.1: Suppose that random variable $X:\Omega\to(0,\infty)$ is continuous and

$h_2(x)=\frac{2e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}}{(1-\theta)\left(1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)^{-2}}$ for $x\varepsilon(0,\infty)$.The pdf of X is (7) provided functions $p\ and\ h_1$ satisfy equation

$$\frac{p'(t)}{p(t)h_2(t)-h_1(t)}=(1-\theta)\left(\alpha+\beta x+\gamma x^2\right)\mathrm{e}^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\left(1-\theta\left(1-\mathrm{e}^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)^{-2}.$$

Remarks 3.5.1: The solution of above equation is

$$p(t)=\exp\left(-2\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)\right)\int\left(-h_1(x)(1-\theta)\left(\alpha+\beta x+\gamma x^2\right)e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\left(1-\theta\left(1-e^{-\left(\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3\right)}\right)\right)^{-2}\right)dx+D$$ where D is constant.

4. CONCLUDING REMARKS

We presented characterizations of QHR-G distribution through hazard function, Mills ratio, reverse hazard function, Elasticity function and ratio of truncated moments.

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Appendix A

Theorem 1: Suppose that probability space $(\Omega, \mathrm{F}, \mathrm{P})$ and interval $[d_1, d_2]$ with $d_1 < d_2$ ($d_1 = -\infty, d_2 = \infty$) are given. Let continuous random variable $X : \Omega \to [d_1, d_2]$ has distribution function F. Let real functions h_1 and h_2 be continuous on $[d_1, d_2]$ such that $\frac{E[h_1(X)/X \geq x]}{E[h_2(X)/X \geq x]} = p(x)$ is real function in simple form. Assume that $h_1,\ h_2\,\varepsilon C([d_1,d_2]),\ p(x)\,\varepsilon C^2([d_1,d_2])$ and F is two times continuously differentiable and strictly monotone function on $[d_1, d_2]$: As a final point, assume that the equation $h_2 p(x) = h_1$ has no real solution in $[d_1, d_2]$. Then $F(x) = \int_{\ln k}^{x} K \left| \frac{p'(t)}{p(t)h_2(t) - h_1(t)} \right| \exp(-s(t))\,dt$ is obtained from the functions h_1, h_2, $p(t)$ and $s(t)$, where $s(t)$ is obtained from equation $s'(t) = \frac{p'(t)h_2(t)}{p(t)h_2(t) - h_1(t)}$ and K is a constant, picked to make $\int_{d_1}^{d_2} dF = 1.$

Remarks

(a)The interval $[d_1, d_2]$ need not be necessarily close in Theorem 4.1.

(b) The function $p(x)$ should be in simple form.