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CHARACTERIZATION OF CERTAIN UNIVARIATE CONTINUOUS DISTRIBUTIONS THROUGH LORENZ CURVE

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ABSTRACT- The Lorenz curve is a display of distributional inequality of a quantity. The Lorenz curve is a function of the cumulative percentage of ordered values plotted onto the corresponding cumulative percentage of their magnitude. The Lorenz curve for a probability distribution is continuous function. The Lorenz curve is a display of function L(G) with plotting cumulative percentage of people G along horizontal axis and cumulative percentage of total wealth L along vertical axis. In this paper, certain univariate continuous distributions are characterized through Lorenz curve.

Key Words: Characterization, Lorenz curve, Inequality

1. INTRODUCTION

The Lorenz curve is mostly applied for the display of the inequality of wealth or income or size in Economics and Ecology. The Lorenz curve was introduced by Lorenz (1905). Distributional inequality of a quantity is displayed by Lorenz Curve. Gini coefficient is numerical measure of information about distributional inequality of a quantity. The Lorenz curve is presentation of the cumulative income distribution. The Lorenz curve is straight line representing equality. Any departure from the straight line indicates inequality. Both Horizontal axis and Vertical axis are in percentages.

Lorenz curve $L\left(G\left(x\right)\right)$ for probability distribution having cumulative distribution function G(x) and probability density function g(x) using Partial Moments is defined as

$$L(\mathrm{G}(x))=\frac{1}{\mu}\int_{-\infty}^{x} y\,dG(y),\ \ where\ \mu=\int_{-\infty}^{\infty} y g(y)dy. \tag{1}$$

Lorenz curve $L\left(G\left(x\right)\right)$ for probability distribution having cumulative distribution function G(x) and probability density function g(x) using quantile function (Gaswirth (1971)) is defined as

$$L(p)=\frac{\int_0^p q(t)dt}{\int_0^1 q(t)dt}=\frac{1}{\mu}\int_0^p q(t)d,\ where\ x=G^{-1}(p)\ and\ \mu=\int_0^1 q(t)dt. \tag{2}$$

1.1 Properties of Lorenz Curve

The Starting and ending points of Lorenz curve are (0, 0) and (1, 1) respectively. Lorenz curve is continuous on [0, 1]. Only for finite Mean, Lorenz curve exists. Lorenz curve is an increasing convex function i.e. $L^{/}(0+)\geq 0$ and its second order derivative is positive (convex) i.e. $L^{//}(p)\geq 0$. The Lorenz curve with positive scaling is invariant. The graph and value of Lorenz curve is always at most distribution function. The convex hull of the Lorenz Curve collapses to the equalitarian line, if there is perfect equality.

2. CHARACTERIZATION

In order to develop a stochastic function for a certain problem, it is necessary to know whether function fulfills the theory of specific underlying probability distribution, it is required to study characterizations of specific probability distribution. Different characterization techniques have developed.
The rest of paper is composed as follows. Characterization of certain univariate continuous distributions is studied through Lorenz curve.

2.1 Characterization of Univariate Continuous Distributions through Lorenz Curve

Theorem 2.1(Sarabia; 2008**)**

Suppose that Lorenz curve $L(p)$ is increasing convex function with finite mean and $L^{//}(p)\geq 0$ exists in (x_1, x_2), then finite positive probability density function $g(x)=\dfrac{1}{\mu L^{//}(G(x))}$ in the interval $\left(\mu L^{/}(x_1^{+}), \mu L^{/}(x_2^{+})\right)$ is obtained from the cumulative distribution function $G(x)$.

Proof

For probability distribution, Lorenz curve $L(G(x))$ from (1) is $L(\mathrm{G}(x))=\dfrac{1}{\mu}\int_{-\infty}^{x} y\,dG(y)$,

After twice differentiation of above equation and simplification we obtain as $\mu L^{//}(G(x))g(x)=1$,

Then probability density function $g(x)$ is $g(x)=\dfrac{1}{\mu L^{//}(G(x))}$ in the interval $\left(\mu L^{/}(x_1^{+}), \mu L^{/}(x_2^{+})\right)$.

3. Univariate Probability Distribution

In this section, Gamma distribution, Beta distribution, Power distribution, Exponential distribution, Pareto distribution, Chi-square, Skew Normal distribution, Folded t-distribution are Characterized through Lorenz curve.

3.1 Gamma distribution

Theorem 3.1: For continuous random variable X having $Gamma(n,\lambda)$, Lorenz curve is $L(p)=\frac{1}{\mu\lambda}\left[nG(t)-tg(t)\right]$ is,

$where\ p=G(t)\ and\ t=G^{-1}(p)$ provided probability density function is $g(t)=\frac{1}{\Gamma(n)}\lambda^n t^{n-1}e^{-\lambda t},\ \lambda>0, \mathrm{t}>0$.

Proof

For continuous random variable X having $Gamma(n,\lambda)$, Lorenz curve for gamma distribution having pdf

$$g(x)=\frac{1}{\Gamma(n)}\lambda^n x^{n-1}e^{-\lambda x}\quad x>0 \text{ is calculated as } L(p)=\frac{1}{\mu}\int_0^t xg(x)dx,$$

$$L(p)=\frac{1}{\mu}\int_0^t x\frac{1}{\Gamma(n)}\lambda^n x^{n-1}e^{-\lambda x}dx=\frac{1}{\mu\lambda}\left[nG(t)-tg(t)\right],$$

$$L(p)=\frac{1}{\mu\lambda}\left[nG(t)-tg(t)\right]. \tag{3}$$

Conversely

Differentiate equation (3), we have $\mu L'(G(t))g(t)=\frac{1}{\lambda}\left[ng(t)-g(t)-tg'(t)\right],$

$$\mu L'(G(t))g(t)=\left[-\left(\frac{1}{\lambda}\frac{1}{\Gamma(n)}\lambda^n t^{n-1}e^{-\lambda t}+\frac{t}{\lambda}\frac{1}{\Gamma(n)}\lambda^n t^{n-1}e^{-\lambda t}\left[\frac{n-1}{t}-\lambda\right]\right)+\frac{n}{\lambda}\frac{1}{\Gamma(n)}\lambda^n t^{n-1}e^{-\lambda t}\right]=tg(t),$$

After simplification we obtain $\mu L'(G(t))=t.$

Again differentiating above equation, we have $\mu L''(G(t))g(t)=1.$

Then probability density function is $g(t)=\frac{1}{\mu L''(G(t))}=\frac{1}{\Gamma(n)}\lambda^n t^{n-1}e^{-\lambda t}$.

$g(t)=\frac{1}{\Gamma(n)}\lambda^n t^{n-1}e^{-\lambda t}\ ,\lambda>0, \mathrm{t}>0$ is probability density function of Gamma distribution.

3.2 Beta distribution

Theorem 3.2: For continuous random variable $X \sim Beta(\mathrm{m},\mathrm{n})$, Lorenz curve is $L(p)=\frac{1}{\mu}\left[\frac{t(t-1)}{m+n}g(t)+\frac{m}{n+m}G(t)\right]$ is,

$where\ p=G(t)\ and\ t=G^{-1}(p)$ provided probability density function is $g(t)=\frac{t^{m-1}(1-t)^{n-1}}{B(m,n)},\quad 0<t<1.$

Proof

For continuous random variable $X \sim Beta(\mathrm{m,n})$, Lorenz curve for beta distribution having pdf

$$g(x)=\frac{x^{m-1}(1-x)^{n-1}}{B(m,n)},\ 0<x<1 \text{ is calculated as } L(p)=\frac{1}{\mu}\int_0^t xg(x)dx,$$

$$L(p)=\frac{1}{\mu}\int_a^t x\frac{x^{m-1}(1-x)^{n-1}}{B(m,n)}dx=\frac{1}{\mu}\int_0^t\frac{x^{m}(1-x)^{n-1}}{B(m,n)}dx,$$

$$L(p)=\frac{1}{\mu}\left[-\frac{t(1-t)}{n}\frac{t^{m-1}(1-t)^{n-1}}{B(m,n)}+\frac{m}{n}\left(\int_0^t\frac{x^{m-1}(1-x)^{n-1}}{B(m,n)}dx-\int_0^t x\frac{x^{m-1}(1-x)^{n-1}}{B(m,n)}dx\right)\right],$$

$$L(p)=\frac{1}{\mu}\left[\frac{t(t-1)}{m+n}g(t)+\frac{m}{n+m}G(t)\right]. \qquad (4)$$

Conversely

Differentiating both sides equation (4),we have $\mu L^{/}(p)g(t)=\left[\frac{m}{n+m}g(t)-\left(\frac{t(1-t)}{n+m}g(t)\right)^{/}\right],$

Using $g^{/}(t)=g(t)\left[\frac{(m-1)}{t}-\frac{(n-1)}{1-t}\right]$, we have

$$\mu L^{/}(G(t))g(t)=\left[\frac{m}{n+m}g(t)-g(t)\left(\frac{(1-t)}{n+m}-\frac{t}{n+m}+\frac{t(1-t)}{n+m}\frac{(m-1)}{t}-\frac{t(1-t)}{n+m}\frac{(n-1)}{1-t}\right)\right]=tg(t).$$

After simplification we obtain $\mu L^{/}(G(t))=t$.

Again differentiating above equation, we have $\mu L^{//}(G(t))g(t)=1$.

Then probability density function is $g(t)=\frac{1}{\mu L^{//}(p)}=\frac{t^{m-1}(1-t)^{n-1}}{B(m,n)},$

$g(t)=\frac{t^{m-1}(1-t)^{n-1}}{B(m,n)},\ m>0, n>0$ is probability density function of Beta distribution.

3.3 Power distribution

Theorem 3.3: For continuous random variable $X \sim Power(\alpha)$, Lorenz curve is $L(p)=\left(1+\frac{1}{\alpha}\right)^{-1}\frac{t^{\alpha+1}}{\mu}$ is,

$where\ p=G(t)\ and\ t=G^{-1}(p)$ provided probability density function is $g(t)=\alpha t^{\alpha-1}, \quad 0<t<1.$

Proof

For continuous random variable $X \sim Power(\alpha)$, Lorenz curve for Power distribution having pdf

$g(x)=\alpha x^{\alpha-1}, \quad 0<x<1$ is calculated as $L(p)=\frac{1}{\mu}\int_0^t xg(x)dx$,

$$L(p)=\frac{1}{\mu}\int_0^t x\alpha x^{\alpha-1}dx=\frac{1}{\mu}\left(1+\frac{1}{\alpha}\right)^{-1}t^{\alpha+1},$$

$$L(p)=\left(1+\frac{1}{\alpha}\right)^{-1}\frac{t^{\alpha+1}}{\mu}. \tag{5}$$

Conversely

Differentiate equation (5), we have $\mu L^{/}(G(t))g(t)=\alpha t^{\alpha}=tg(t)$,

After simplification we obtain $\mu L^{/}(G(t))=t$,

Again differentiating above equation, we have $\mu L^{//}(G(t))g(t)=1.$

Then probability density function $g(x)$ is $g(t)=\frac{1}{\mu L^{//}(p)}=\alpha t^{\alpha-1}$,

$g(t)=\alpha t^{\alpha-1} \quad 0<t<1$ is probability density function of Power distribution.

3.4 Exponential distribution

Theorem 3.4: For continuous random $X \sim Exp(\lambda)$, Lorenz curve is $L(p)=\frac{1}{\mu}\left[\frac{1}{\lambda}\left(1-e^{-\lambda t}\right)-te^{-\lambda t}\right]$ is,

$where\ p=G(t)\ and\ t=G^{-1}(p)$ provided probability density function is $g(x)=\lambda e^{-\lambda t}, \quad t,\lambda>0$.

Proof

For continuous random variable $X \sim Exp(\lambda)$, Lorenz curve for Exponential distribution having pdf

$g(x)=\lambda e^{-\lambda x},\ x,\lambda>0$ is calculated as $L(p)=\frac{1}{\mu}\int_0^t xg(x)dx=\frac{1}{\mu}\int_0^t x\lambda e^{-\lambda x}dx$

$$L(p)=\frac{1}{\mu}\left[\frac{1}{\lambda}\left(1-e^{-\lambda x}\right)-te^{-\lambda t}\right]. \tag{6}$$

Conversely

Differentiating both sides of equation (6), we have $\mu L^{/}(p)g(t)=\left[\lambda te^{-\lambda t}\right]=tg(t)$,

After simplification we obtain $\mu L^{/}(p)=t$,

Again differentiating above equation, we have $\mu L^{//}(G(t))g(t)=1$.

Then probability density function is $g(t)=\frac{1}{\mu L^{//}(p)}=\lambda e^{-\lambda t}$,

$g(x)=\lambda e^{-\lambda x},\ x,\lambda>0$ is probability density function of Exponential distribution.

3.5 Pareto distribution

Theorem 3.5: For continuous random variable $X \sim Pareto(\alpha)$, Lorenz curve is $L(p)=\frac{1}{\mu}\frac{\alpha}{\alpha-1}\left[1-t^{1-\alpha}\right]$ is,

$where\ p=G(t)\ and\ t=G^{-1}(p)$ provided probability density function is $g(t)=\alpha t^{-\alpha-1},\ 1<t<\infty$.

Proof

For continuous random variable $X \sim Pareto(\alpha)$, Lorenz curve for Pareto distribution having pdf

$g(t)=\alpha t^{-\alpha-1},\ 1<t<\infty$ is calculated as $L(p)=\frac{1}{\mu}\int_0^t xg(x)dx$,

$$L(p)=\frac{1}{\mu}\int_1^t \alpha x^{-\alpha}dx,$$

$$L(p)=\frac{1}{\mu}\frac{\alpha}{\alpha-1}\left(1-t^{1-\alpha}\right). \tag{7}$$

Conversely

Differentiating both sides of equation (7), we have $\mu L^{/}(p)g(t)=\mu L^{/}(G(t))g(t)=t\left(\alpha t^{-\alpha-1}\right)=tg(t)$, After simplification we obtain $\mu L^{/}\left(G(t)\right)=t$,

Again differentiating above equation, we have $\mu L^{//}\left(G(t)\right)g(t)=1$,

Then probability density function is $g(t)=\dfrac{1}{\mu L^{//}\left(G(t)\right)}=\alpha t^{-\alpha-1}$,

$g(t)=\alpha t^{-\alpha-1},\ 1<t<\infty$ is probability density function of Pareto distribution.

3.6 Chi-square distribution

Theorem 3.6: For continuous random $X \sim Chi(k)$, Lorenz curve is $L(p)=\dfrac{1}{\mu}\left[kG(t)-2tg(t)\right]$ is, $where\ p=G(t)\ and\ t=G^{-1}(p)$ provided probability density function is $g(t)=\dfrac{2^{-k/2}e^{-t/2}t^{k/2-1}}{\Gamma(k/2)}\quad t>0$.

Proof

For continuous random variable $X \sim Chi(k)$, Lorenz curve for Chi-square distribution having pdf

$g(t)=\dfrac{2^{-k/2}e^{-t/2}t^{k/2-1}}{\Gamma(k/2)}\quad t>0$ is calculated as $L(p)=\dfrac{1}{\mu}\int_0^t xg(x)dx=\dfrac{1}{\mu}\int_0^t x\dfrac{2^{-k/2}e^{-x/2}x^{k/2-1}}{\Gamma(k/2)}dx$,

$$L(p)=\frac{1}{\mu}\left[kG(t)-2tg(t)\right]. \qquad (8)$$

Conversely

Differentiate equation (8), we have $\mu L^{/}(p)g(t)=\mu L^{/}\left(G(t)\right)g(t)=\left[-\left(2g(t)+2tg^{/}(t)\right)+kg(t)\right]$,

$$\mu L^{/}\left(G(t)\right)g(t)=\left[-\left(2\frac{2^{-k/2}e^{-t/2}t^{k/2-1}}{\Gamma(k/2)}+2\frac{2^{-k/2}t\left[-\frac{1}{2}e^{-t/2}t^{k/2-1}+\left(k/2-1\right)e^{-t/2}t^{k/2-2}\right]}{\Gamma(k/2)}\right)+k\frac{2^{-k/2}e^{-t/2}t^{k/2-1}}{\Gamma(k/2)}\right],$$

$$\mu L^{/}\left(G(t)\right)g(t)=g(t)\left[-2\left(1-\frac{1}{2}t+\left(k/2-1\right)\right)+k\right]=tg(t),$$

After simplification we obtain $\mu L^{/}\left(G(t)\right)=t$,

Again differentiating above equation, we have $\mu L^{//}\left(G(t)\right)g(t)=1$, Then probability density function is $g(t)=\frac{1}{\mu L^{//}(G(t))}=\frac{2^{-k/2}e^{-t/2}t^{k/2-1}}{\Gamma(k/2)}$,

$g(t)=\frac{2^{-k/2}e^{-t/2}t^{k/2-1}}{\Gamma(k/2)}$ $\quad t>0$ is probability density function of Chi-square distribution.

3.7 Skew Normal distribution

Theorem 3.7:: For continuous random $X \sim SkewNormal(\lambda)$, Lorenz curve is $L(p)=\frac{1}{\mu}[t\Phi(t,\lambda)-\mathrm{H}(t,\lambda)]$ is,

$where\ p=G(t)\ and\ t=G^{-1}(p)$ provided probability density function is $g(t)=2\phi(t)\Phi(\lambda t),\ -\infty<t<\infty.$.

Proof

For continuous random variable $X \sim SkewNormal(\lambda)$, Lorenz curve for Skew Normal distribution having pdf $g(x)=2\phi(x)\Phi(\lambda x)$ and distribution function $\Phi(x,\lambda)=[\Phi(x)-2T(x,\lambda)]$ is calculated as

$$L(p)=\frac{1}{\mu}\int_0^t xg(x)dx=\frac{1}{\mu}\int_0^t x2\phi(x)\Phi(\lambda x)dx=\frac{1}{\mu}[t\Phi(t,\lambda)-\mathrm{H}(t,\lambda)],$$

$$L(p)=\frac{1}{\mu}[t\Phi(t,\lambda)-\mathrm{H}(t,\lambda)]. \tag{9}$$

Conversely

Differentiating both sides of equation (9), we have

$\mu L^{/}(\mathrm{p})\,\mathrm{g}(t)=\{\Phi(\mathrm{t},\lambda)+2x\phi(t)\Phi(\lambda t)-\Phi(\mathrm{t},\lambda)\}=t(2\phi(t)\Phi(\lambda t))=tg(t)$,

After simplification we obtain $\mu L^{/}(p)=t$,

Again differentiating above equation, we have $\mu L^{//}(p)g(t)=L^{//}(G(t))g(t)=1$,

Then probability density function is $g(t)=\frac{1}{\mu L^{//}(p)}=2\phi(t)\Phi(\lambda t)$,

$g(t)=2\phi(t)\Phi(\lambda t),\ -\infty<t<\infty.$ is probability density function of Skew Normal distribution.

3.8 The Folded t-distribution

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Theorem 3.8: For continuous random $X \sim Folded\ t-\ distribution\ with\ df = 2$, Lorenz curve is

$L(p)=\frac{1}{\mu\sqrt{2}}\left[1-\left(1+\left(\frac{t}{\sqrt{2}}\right)^2\right)^{-\frac{1}{2}}\right]$ is, $where\ p=G(t)\ and\ t=G^{-1}(p)$ provided probability density function is

$$g(t)=\frac{2}{\left(2+t^2\right)^{\frac{3}{2}}},\ t>0.$$

Proof

For continuous random variable $X \sim Folded\ t-\ distribution\ with\ df = 2$, Lorenz curve for Folded t-distribution distribution having pdf $g(x)=\frac{2}{\left(2+x^2\right)^{\frac{3}{2}}}$ is calculated as

$$L(p)=\frac{1}{\mu}\int_0^t xg(x)dx=\frac{1}{\mu}\int_0^t x\frac{2}{\left(2+x^2\right)^{\frac{3}{2}}}dx=\frac{2}{\mu}\left[\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{\left(2+t^2\right)}}\right],$$

$$L(p)=\frac{1}{\mu\sqrt{2}}\left[1-\left(1+\left(\frac{t}{\sqrt{2}}\right)^2\right)^{-\frac{1}{2}}\right]. \qquad (10)$$

Conversely

Differentiating both sides of equation (10), we have $\mu L'(p)g(t)=t\left(2+t^2\right)^{-\frac{3}{2}}=tg(t)$,

After simplification we obtain $\mu L'(p)=\mu L'(G(t))=t$,

Again differentiating above equation, we have $\mu L''(G(t))g(t)=1$,

Then probability density function $g(x)$ is $g(t)=\frac{1}{\mu L''(G(t))}=\frac{2}{\left(2+t^2\right)^{\frac{3}{2}}}$,

$g(t)=\frac{2}{\left(2+t^2\right)^{\frac{3}{2}}},\ t>0.$ is probability density function of Folded t- distribution.

4. Concluding Remarks

In this research, we presented characterization of Gamma distribution, Beta distribution, Power distribution, Exponential distribution, Pareto distribution, Chi-square, Skew Normal distribution and Folded t-distribution through Lorenz curve.

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