New Fuzzy Divergence Measure and its Applications: A New Approach

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Abstract- Various authors and researchers have established fuzzy divergence measures and their applications in multi-criteria decision making (MCDM), pattern recognition (PR), medical diagnosis (MD), fuzzy clustering, speech recognition etc. Here, we have derived a new fuzzy divergence measure and investigated their properties to existence and validity. Also investigated fuzzy divergence measure in PR, MCDM and MD. Compared the established results by various authors and researcher. Novelty of this research may be useful to industries for decision making, identifying the medical diseases and pattern recognition.

Key Words: Entropy, fuzzy sets, new f-divergence measure, MCDM, PR, MD. Mathematics Subject Classification (2010) 94A17, 03E72, 68T10, 90B50, 92C50.

Introduction
In information theory, Shannon [20] defined the entropy measure for a probability distribution. Fuzzy entropy, a fuzziness measure (cf.[13]) often used and cited in many literatures, was introduced by Zadeh [27, 28]; then De Luca and Termini [14] defined an entropy of a fuzzy set based on Shannon’s function. Later on, many other researchers made more effort in this particular area. In 1975, Kaufmann [11] proposed a new fuzziness measure of fuzzy set by a distance between its membership function and its nearest classical(Ordinary) sets. Entropy, is a very important notion for measuring uncertain information and received great attention in the past decades. It has broad applications in many areas such as Pattern recognition, decision making, medical diagnosis, signal and image processing, fuzzy clustering, speech recognition, feature selection, fuzzy aircraft control, bio-informatics etc.

Afterwards, a number of other researchers have studied the fuzzy diver-gence measures in different ways and provide their applications in different areas. Fuzzy divergence measure introduced by Fan and Xie [5] is based on exponential operation and its relation with fuzzy divergence measure [1]. Prakash et al. [17] proposed two fuzzy divergence measures corresponding to Ferreri [7] probabilistic measure of directed divergence. Ghosh et al. [8] gave its application in the area of automated leukocyte recognition. The study submitted by Montes et al. [15] in 2002 was based on special classes of divergence measures and used the link between fuzzy and probabilistic uncertainty. Bhatia and Singh [2] proposed the fuzzy divergence measure corresponding to Taneja Tomar and Ohlan [22] studied a sequence of fuzzy mean di erence di-vergence measures by establishing inequalities among them and provided their applications in the context of consistency in linguistic variables and pattern recognition. Tomar and Ohlan [24] introduced a new parametric generalized exponential fuzzy divergence measure corresponding to Verma and Sharma [25] with its application to strategic decision making.

In recent times, the literature on applications of information and divergence measures between fuzzy sets has extended considerably, still there is a possibility of developing better divergence measures can be developed which can be applied to various elds. Here we study a new symmetric generalized measure of fuzzy divergence and its essential properties to check its authenticity. The new generalized measure has elegant properties which are proven in the paper to present the efficiency of the proposed measure.

This paper is organized as follows. In Section 1, some important concepts of probability theory will be reviewed along with the illustration of fuzzy set theory using the membership degree and non-membership degree of fuzzy set. In Section 2, a new symmetric fuzzy divergence measure between fuzzy sets is proposed. Section 3 provides some more elegant properties of the proposed measure.
in a number of theorems. Finally, the applications of proposed generalized fuzzy divergence measure to MCDM, PR and MD are illustrated by three numerical examples in Section 4. In section 5, results and discussion. Finally, some concluding remarks are drawn in Section 6.

1.1 Shannon Entropy

Shannon firstly use the word entropy to measure an uncertain degree of the randomness in a probability distribution and defined the information contained in an experiment. It is given by

\[ H(P) = \sum_{i=1}^{n} p_i \log p_i \]  

(1.1)

Directed divergence measure is a relative entropy measure which provides a distance formula between the two discrete probability distributions. Kullback and Leibler [12] firstly proposed a measure of directed divergence between the two distributions \( P = (p_1, p_2, p_3, \ldots, p_n) \) and \( Q = (q_1, q_2, q_3, \ldots, q_n) \) as:

\[ D(P \mid Q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} \]

The New f-divergence contains several divergences used in determining the affinity/distance between two probability distributions. This divergence is introduced by Jain and Saraswat and is given by

\[ S_f(P; Q) = \sum_{i=1}^{n} q_i f \left( \frac{p_i}{q_i} \right) \]

It is well known that \( S_f(P, Q) \) is a multi-purpose functional form which results in number of general divergence measures. Most common choices of \( f \) satisfy \( f(1) = 0 \), so that \( S_f(P, Q) = 0 \). Convexity ensures that divergence measure \( S_f(P, Q) \) is non-negative.
1.2 Fuzzy Concepts

Fuzziness, a feature of uncertainty, results from the lack of sharp difference of being or not being a member of the set, i.e., the boundaries of the set under consideration are not sharply defined. A fuzzy set A defined on a universe of discourse X is given by Zadeh and is defined below:

\[ A = \{ x \in X | \mu_A(x) \} \]

where \( \mu_A(x) : X \rightarrow [0, 1] \) is the membership function of A. The membership value describes the degree of belongingness of X in A. When valued in [0, 1], it is the characteristic function of crisp (i.e. non fuzzy) set. Zadeh [28, 27] gave some notions related to fuzzy sets, some of which are listed below:

1. Complement: \( A^c = \{ x \in X | 1 - \mu_A(x) \} \) for all \( x \in X \).
2. Union: \( A \cup B = \{ x \in X | \max(\mu_A(x), \mu_B(x)) \} \) for all \( x \in X \).
3. Intersection: \( A \cap B = \{ x \in X | \min(\mu_A(x), \mu_B(x)) \} \) for all \( x \in X \).

Considering the concept of fuzzy sets, De Luca and Termini [14] introduced the measure of fuzzy entropy corresponding to Shannon’s entropy and given as:

\[
H(N) = \sum_{i=1}^{n} \mu_A(x_i) \log_2 \mu_A(x_i) + (1 - \mu_A(x_i)) \log_2 (1 - \mu_A(x_i))
\]

This idea of divergence measure was extended from probabilistic to fuzzy set theory by Bhandari and Pal [1] by giving a fuzzy information measure for discrimination of a fuzzy set B relative to some other fuzzy set A. Let A and B be two fuzzy sets defined in discrete universe of discourse \( X = \{ x_1, x_2, ..., x_n \} \), having the membership values \( \mu_A(x_i) \) and \( \mu_B(x_i) \), i = 1, 2,..., n respectively, then the fuzzy divergence measure of fuzzy set B relative to A is given by:

\[
D(A \cup B) = \sum_{i=1}^{n} \mu_A(x_i) \log_2 \frac{\mu_A(x_i)}{\mu_A(x_i)} + \frac{1}{\mu_A(x_i)} \cdot \mu_B(x_i)
\]

Satisfying the condition

\[
D(A, B) = 0, \quad D(A, B) = 0 \quad \text{if} \quad A = B,
\]

D(A, B) is a convex function of \( \mu_A(x_i) \).
If \( X \) is a universe of discourse and \( F(X) \) is the set of all fuzzy subsets, a mapping \( L : F(X) \times F(X) \rightarrow \mathbb{R} \) is a fuzzy divergence measure if and only if for each \( A, B, C \in F(X) \), the following axioms hold:

\[ \begin{align*}
(a_1) \quad & L(A, B) = 0, \\
(a_2) \quad & L(A, A) = 0, \\
(a_3) \quad & \max_{A \cup C, B \cup C} L(A \setminus C, B \setminus C) \geq L(A, B).
\end{align*} \]

Non-negativity of \( L(A, B) \) is the natural assumption.

**Generalized Symmetric fuzzy divergence measure**

Here, we shall propose a symmetric generalized measure of divergence between two fuzzy sets \( A, B \) of universe of discourse \( X = \{x_1, x_2, x_3, \ldots, x_n\} \) having the membership values \( 2 (0, 1) \) corresponding to [19] and is given by

\[
N_t(A, B) = \sum_{i=1}^{2n+1} \left( \begin{array}{c}
A(x_i) \times B(x_i) \\
1 - \left( A(x_i) + B(x_i) \right) \\
X
\end{array} \right)^{2t} + \left( 1 - A(x_i) \right) \left( 1 - B(x_i) \right)^{2t}
\]

(2.1)

\( t = 0, 1, \ldots \)

**2.1 Theorem**

\( N_t(A, B) \) is a valid measure of fuzzy divergence.

Proof. It is clear from (2.1) that

\[ \begin{align*}
N_t(A, B) \quad & 0 \\
N_t(A, B) \quad & = 0 \text{ if } A(x_i) = B(x_i), \\
\max_{A \setminus C, B \setminus C} N_t(A \setminus C, B \setminus C) \geq N_t(A, B)
\end{align*} \]

We divide \( X \) into six subsets:

\[ \begin{align*}
X_1 & = \{x \in X \mid \lambda(x) < \mu(x) \} \\
X_2 & = \{x \in X \mid \lambda(x) < c(x) \} \\
X_3 & = \{x \in X \mid \mu(x) < \lambda(x) \} \\
X_4 & = \{x \in X \mid \mu(x) < c(x) \} \\
X_5 & = \{x \in X \mid c(x) < \lambda(x) \} \\
X_6 & = \{x \in X \mid c(x) < \mu(x) \}
\end{align*} \]

[35] www.ijergs.org
\[ X_6 = \frac{fx}{2} X, \quad c(x) < b(x) < \lambda(x) g \]

In set \( X_1 \),

\[ A \cup C = \text{Union of } A \text{ and } B \quad A\cup C(x) = \max f_A(x), \quad c(x) g = c(x) \]

\[ B \cup C = \text{Union of } B \text{ and } C \quad B\cup C(x) = \max f_B(x), \quad c(x) g = c(x) ; \]

\[ A \setminus C = \text{intersection of } A \text{ and } C \quad A\setminus C(x) = \min f_A(x), \quad c(x) g = \lambda(x); \]

\[ B \setminus C = \text{intersection of } B \text{ and } C \quad B\setminus C(x) = \min f_B(x), \quad c(x) g = \mu(x); \]

\[ N_i(A \cup B, C) = N_i(C, C) = 0, \]

\[ N_i(A \setminus B, C) = N_i(A, B), \]

So, \( \max f N_i(A \cup B, C), N_i(A \setminus B, C) g = N_i(A, B) \).

Similarly, in the sets \( X_2, X_3, X_4, X_5, X_6 \) we have

\[ \max f N_i(A \cup C, B \cup C), N_i(A \setminus C, B \setminus C) g = N_i(A \cup B) \]

Thus, \( \max f N_i(A \cup C, B \cup C), N_i(A \setminus C, B \setminus C) g N_i(A, B) \) for all \( A, B, C \) \( X \)

**Properties of proposed fuzzy divergence measure**

In this section we provide properties of the proposed generalized fuzzy divergence measure (2.1) in accordance with the following theorems. To prove these theorems we assume that \( X \) divided into two parts \( X_1 \) and \( X_2 \) such that the sets

\[ X_1 = \frac{fx}{2} X, \quad \lambda(x_i) < b(x_i) g \]

\[ X_2 = \frac{fx}{2} X, \quad \lambda(x_i) < b(x_i) g \]

Theorem 1. (a) \( N_i(A \cup B) = N_i(B, A \cup B) \).

\( N_i(A \cup B, A) + N_i(A \setminus B, A) = N_i(B, A) \).

\( N_i(A \cup B, A \setminus B) = N_i(A, B) \).

(d) \( N_i(A \cup B, C) + N_i(A \setminus B, C) = N_i(A, B) + N_i(B, C) \).

Theorem 2. (a) \( N_i(A, A) = N_i(A, A) \) \( - \) \( - \)

\( N_i(A, B) = N_i(A, B) \) \( - \) \( - \)

\( N_i(A, B) = N_i(A, B) \) \( - \) \( - \)

\( N_i(A, B) + N_i(A, B) = N_i(A, B) + N_i(A, B) \) \( - \) \( - \) \( - \) \( - \) \( - \) \( - \) \( - \)

**Applications of Proposed fuzzy divergence measure**

Here we are going to introduce the application in the context of MCDM, PR and MD.
4.1 Multi-Criteria Decision Making

Decision-making deals with the problem of choosing the best alternative with the highest degree of satisfaction for all the appropriate criteria or goals. Multi-criteria decision-making (MCDM) is the most well known branch of decision-making that allows decision-makers to rank and select alternatives according to different criteria. Using below method we can solve the MCDM problems with the help of proposed generalized fuzzy divergence measure.

Let \( R = (R_1, R_2, R_3, \ldots, R_m) \) be a set of options, \( D = (D_1, D_2, D_3, \ldots, D_m) \) be the set of criteria. The Characteristics of the option \( R_i \) in term of criteria \( D \) are represented by the following FSs:

\[
R_i = f < D_{ij}, \ldots, < n_g >, \quad i = f 1, 2, 3, \ldots, m \quad \text{and} \quad j = f 1, 2, 3, \ldots, n_g
\]

where \( ij \) indicates the degree that the option \( R_i \) satisfies the criteria \( D_j \).

We can solve the above MCDM problem using Eq. (2.1).

Step 1: Find out the positive-ideal solution \( R^+ \) and negative-ideal solution \( R^- \):

\[
R^+ = f < 1^+, \ldots, < n^+ >, \quad R^- = f < 1^-, \ldots, < n^+ >
\]

where for each \( j = 1, 2, 3, \ldots, n \).

\[
\begin{align*}
 j^+ &= \max_i \ ij \\
 j^- &= \min_i \ ij
\end{align*}
\]

Step 2: Calculate \( N_i(R^+, R_i) \) and \( N_i(R^-, R_i) \) using Eq. (2.1)

Step 3: Calculate the relative fuzzy divergence measure \( N_i(R_i) \) of alternative \( R_i \) with respect to \( R^+ \) and \( R^- \), where

\[
N_i(R^+; R_i) = N_i(R^-; R_i) + N_i(R^+; R_i), \quad i = 1, 2, \ldots, m.
\]

Step 4: Rank the preference order of all alternatives according to the relative fuzzy divergence measure.

Step 5: Select the best alternative \( R_k \) with the smallest \( N_i(R_k) \).

To solve a real problem related to MCDM, we establish the applicability of new fuzzy divergence measure. For this we are going to consider customer decision-making problem given below.

Case study Consider a customer who wants to buy a car. Let there be 5 types of cars : the alternatives \( R = f R_1, R_2, R_3, R_4, R_5 \) be available in the market. To buy a car the customer takes the following four criteria : (i) Quality of product \( (D_1) \), (ii) price \( (D_2) \), (iii) Technical capability \( (D_3) \) and (iv) Fuel Economy \( (D_4) \).
The ve possible options are to be evaluated by using decision making un-der the above four criteria in the following form:

\[ R_1 = f <D_1, .5>, <D_2, .3>, <D_3, .4>, <D_4, .7> \]
\[ R_2 = f <D_1, .2>, <D_2, .7>, <D_3, .6>, <D_4, .6> \]
\[ R_3 = f <D_1, .8>, <D_2, .5>, <D_3, .9>, <D_4, .2> \]
\[ R_4 = f <D_1, .6>, <D_2, .4>, <D_3, .7>, <D_4, .5> \]
\[ R_5 = f <D_1, .6>, <D_2, .5>, <D_3, .5>, <D_4, .7> \]

The stepwise computational procedure to solve the above multi-criteria fuzzy decision- problem now goes as follows.

Step 1: The positive-ideal solution (\( R^+ \)) and negative-ideal solution (\( R^- \)) respectively are

\[ R^+ = f <D_1, .8>, <D_2, .7>, <D_3, .9>, <D_4, .7> \]
\[ = f <D_1, .2>, <D_2, .3>, <D_3, .4>, <D_4, .2> \]

Step 2: Table (1) and (2) shows the calculated numerical values of \( N_t(R^+, R_i) \) and \( N_t(R_i, R_j) \) using Eq. (2.1) for \( t \) 0

Step 3: Calculated values of the relative fuzzy divergence measure \( N_t(R_i) \)
for \( i= 1, 2, 3, 4, 5 \) with \( t \) are presented in table (3)

Step 4: According to the calculated numerical values of relative divergence measure for di erent values of \( t \) ranking order of alternative as follows:

For \( t = 1 \), \( R_3 > R_4 > R_5 > R_2 > R_1 \)
For \( t = 5 \), \( R_3 > R_4 > R_5 > R_2 > R_1 \)
For \( t = 10 \), \( R_3 > R_4 > R_5 > R_2 > M_1 \)
For \( t = 100 \), \( R_3 > R_4 > R_5 > R_2 > R_1 \)

Table 1: Calculated numerical values of \( N_t(R^+, R_i) \), \( t \) 0

<table>
<thead>
<tr>
<th></th>
<th>( t=1 )</th>
<th>( t=5 )</th>
<th>( t=10 )</th>
<th>( t=100 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_t(R^+, R_1) )</td>
<td>78309</td>
<td>42980</td>
<td>52645</td>
<td>20.74649</td>
</tr>
<tr>
<td>( N_t(R^+, R_2) )</td>
<td>74554</td>
<td>07039</td>
<td>00396</td>
<td>1.02861e-25</td>
</tr>
<tr>
<td>( N_t(R^+, R_3) )</td>
<td>38793</td>
<td>00864</td>
<td>00014</td>
<td>4.25021e-36</td>
</tr>
<tr>
<td>( N_t(R^+, R_4) )</td>
<td>27570</td>
<td>00082</td>
<td>3.38702e-06</td>
<td>3.88065e-49</td>
</tr>
<tr>
<td>( N_t(R^+, R_5) )</td>
<td>39811</td>
<td>09831</td>
<td>0322</td>
<td>6.11111e-11</td>
</tr>
</tbody>
</table>
Table 2: Calculated numerical values of \( N_t(R, R_i) \), \( t = 0 \)

<table>
<thead>
<tr>
<th></th>
<th>( t=1 )</th>
<th>( t=5 )</th>
<th>( t=10 )</th>
<th>( t=100 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_t(R, R_1) )</td>
<td>0.45944</td>
<td>0.00884</td>
<td>0.0014</td>
<td>4.25021e-36</td>
</tr>
<tr>
<td>( N_t(R, R_2) )</td>
<td>0.44046</td>
<td>0.00192</td>
<td>6.77459e-06</td>
<td>7.76130e-49</td>
</tr>
<tr>
<td>( N_t(R, R_3) )</td>
<td>1.08374</td>
<td>0.47986</td>
<td>5.2961</td>
<td>1.02861e-25</td>
</tr>
<tr>
<td>( N_t(R, R_4) )</td>
<td>0.43525</td>
<td>0.00187</td>
<td>6.77459e-06</td>
<td>7.76130e-49</td>
</tr>
<tr>
<td>( N_t(R, R_5) )</td>
<td>0.60647</td>
<td>0.01031</td>
<td>0.0014</td>
<td>4.25021e-36</td>
</tr>
</tbody>
</table>

Then by table (3) variation in values of \( t \) brings change in ranking, but leaves the best choice unchanged. So \( R_3 \) is the most perfectly alternative.

Table 3: Computed values of relative divergence measure \( N_t(R_i) \) for \( i = 1, 2, 3, 4, 5 \)

<table>
<thead>
<tr>
<th></th>
<th>( t=1 )</th>
<th>( t=5 )</th>
<th>( t=10 )</th>
<th>( t=100 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_t(R_1) )</td>
<td>0.63024</td>
<td>0.97985</td>
<td>0.99970</td>
<td>1</td>
</tr>
<tr>
<td>( N_t(R_2) )</td>
<td>0.62681</td>
<td>0.97345</td>
<td>0.99830</td>
<td>1</td>
</tr>
<tr>
<td>( N_t(R_3) )</td>
<td>0.26359</td>
<td>0.01769</td>
<td>0.00030</td>
<td>4.13199e-11</td>
</tr>
<tr>
<td>( N_t(R_4) )</td>
<td>0.38779</td>
<td>0.30483</td>
<td>0.33330</td>
<td>0.33330</td>
</tr>
<tr>
<td>( N_t(R_5) )</td>
<td>0.39629</td>
<td>0.90581</td>
<td>0.99580</td>
<td>1</td>
</tr>
</tbody>
</table>

Case study by using weight vector

Fuzzy MCDM problems consist of \( m \) alternatives \( a_i (i=1(1)m) \) such that alternative is achieved by means of \( n \) criteria \( b_j (j=1(1)m) \) \( @_{ij} \) is constructed by alternative \( a_i \) with respect to criterion \( b_j \), are fuzzy values (FVs). Let \( w_j \)

\[ w_j \]  

be weight of criterion such that \( w_j \) \( P \) \( w_j = 1, W = (w_1, w_2, ..., w_n)^T \)

\( i=1 \)

It is worth mentioning that proposed method is appropriate for circumstances where the number of decision experts is small such that they assess the criterion based on their experience and knowledge and the alternatives could be of any type, then these assessment of alternatives can be converted to FVs.

If the information regarding the criterion weight vector is not completely known or only partially known, then this proposed method can be used to solve the MCDM problems. This method consists of the following steps:
Step 1: Generate decision matrix $G = (@_{ij})_{m \times n}$

The decision experts furnish all the feasible evaluations regarding the alternative $a_i$ with respect to criterion $b_j$ mentioned by fuzzy values $@_{ij}$ ($i = 1(1)m$, $j = 1(1)n$)

Step 2: Compute PIS and NIS.

The solution which result in optimum value for each criterion is ideal solution. The optimal values (PIS) for diverse criterion are altered and pointed out as

$$\max @_{ij} \text{ or benif itcreterionb}_j <$$

$$i = 1(1)m$$

""" = (4.1)

$$\min @_{ij} \text{ or costcreterionb}_j$$

$$i = 1(1)m$$

for $j = 1(1)n$.

and

$$\max @_{ij} \text{ or benif itcreterionb}_j <$$

$$i = 1(1)m$$

""" = (4.2)

$$\min @_{ij} \text{ or costcreterionb}_j$$

$$i = 1(1)m$$

for $j = 1(1)n$.

Step 3: Compute the weight vector

Overall performance of the alternative $a_i$ computed by given formula

$$k(a_i) = w_j k_{ij}; \text{ where } k_{ij} = \frac{k^{ij}}{k + k}$$

$$= 1$$

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Apparently, the larger value of $k(a_i)$ shows the better the alternative. There fore, all the alternatives are measured as a whole to construct a combined weight vector. Thus, linear programming model is demonstrated as follows:

$$\begin{align*}
\text{max } & \quad K(a_i) = w_i k_{ij}; \\
\text{subject to } & \quad X_i X X
\end{align*}$$

(4.4)
\[ w_s \leq \frac{4}{5}, w_2 - w_5 \leq \frac{3}{5} \text{ such that } w_j = 1 \]

i=1

Step 1: Crisp values in Table 4 are fuzzied by generating the form FSs for each criterion and decision matrix (Table 5) for FSs is constructed.

\[
A_1 = \frac{0.2846}{20.5} + \frac{0.75}{10} + \frac{0.234}{2.21} + \frac{0.23}{2} + \frac{0.24}{4.8}, \quad A_2 = \frac{0.318}{23.31} + \frac{0.248}{2} + \frac{0.766}{24.7} + \frac{0.097}{0.67} + \frac{0.76}{27},
\]

Table 4: Average actual numerical value of criteria

<table>
<thead>
<tr>
<th>i</th>
<th>b_1</th>
<th>b_2</th>
<th>b_3</th>
<th>b_4</th>
<th>b_5</th>
</tr>
</thead>
<tbody>
<tr>
<td>a_1</td>
<td>20.50</td>
<td>10.00</td>
<td>2.21</td>
<td>2.00</td>
<td>4.80</td>
</tr>
<tr>
<td>a_2</td>
<td>23.31</td>
<td>2.00</td>
<td>24.7</td>
<td>.67</td>
<td>27.00</td>
</tr>
<tr>
<td>a_3</td>
<td>60.06</td>
<td>10.00</td>
<td>5.65</td>
<td>2.92</td>
<td>5.5</td>
</tr>
<tr>
<td>a_4</td>
<td>16.86</td>
<td>5.00</td>
<td>9.70</td>
<td>1.25</td>
<td>11.40</td>
</tr>
</tbody>
</table>

\[
A_3 = \frac{0.759}{60.06} + \frac{0.75}{10} + \frac{0.315}{5.65} + \frac{0.32}{2.92} + \frac{0.278}{6.5}, \quad A_4 = \frac{0.241}{16.86} + \frac{0.437}{5} + \frac{0.41}{9.7} + \frac{0.155}{1.25} + \frac{0.394}{11.4}
\]

Step 2: PIS and NIS are calculated by using (4.1) and (4.2) are as follows:

\[
** = f_{0.759,0.75,0.234,0.23,0.24}, \quad ^* = f_{0.241,0.248,0.766,0.32,0.76}
\]

Step 3: Calculating \( k_{ij}^* \) and \( k_{ij}^* \):

\[
k_{11}^* = 0.2919, \quad k_{12}^* = 0.0000, \quad k_{13}^* = 0.0000, \quad k_{14}^* = 0.0000, \quad k_{15}^* =
\]

0.0000,

\[
k_{21}^* = 0.2449, \quad k_{22}^* = 0.3369, \quad k_{23}^* = 0.3948, \quad k_{24}^* = 0.0377, \quad k_{25}^* = 0.3706.
\]
$k_3$

$1^* = 0.0000$, $k_{32}^* = 0.0000$, $k_{33}^* = 0.0084$, $k_{34}^* = 0.0104$, $k_{35}^* = 0.0019$,

$41^* = 0.3667$, $k_{42}^*$ = 0.1151, $k_{43}^* = 0.0376$, $k_{44}^* = 0.0093$, $k_{45}^* = 0.0287$

and

$1^1 0.0025$, $k_{12} = 0.3369$, $k_{13} = 0.3948$, $k_{14} = 0.0104$, $k_{15} = 0.3706$,

$2^1 0.0075$, $k_{22} = 0.0000$, $k_{23} = 0.0000$, $k_{24} = 0.0995$, $k_{25} = 0.0000$.

$3^1 0.3667$, $k_{32} = 0.3369$, $k_{33} = 0.2597$, $k_{34} = 0.0000$, $k_{35} = 0.3039$,

$41 0.0000$, $k_{42} = 0.0421$, $k_{43} = 0.1539$, $k_{44} = 0.0416$, $k_{45} = 0.1619$.

Next, the overall performances, by using (4.3), of alternative are calculated as follows:

$k_{11} = 0.0085$, $k_{12} = 1.0000$, $k_{13} = 1.0000$, $k_{14} = 1.0000$, $k_{15} = 1.0000$, $k_{21} = 0.0297$, $k_{22} = 0.0000$, $k_{23} = 0.0000$, $k_{24} = 0.7252$, $k_{25} = 0.0000$,

$k_{31} = 1.0000$, $k_{32} = 1.0000$, $k_{33} = 0.9687$, $k_{34} = 0.0000$, $k_{35} = 0.9938$, $k_{41} = 0.0000$, $k_{42} = 0.2678$, $k_{43} = 0.8037$, $k_{44} = 0.8173$, $k_{45} = 0.8494$.

Step 4: Construct the model and compute the weight vector.

$max \, k = 1.0382 \, w_1 + 2.2678 w_2 + 2.7784 w_3 + 2.5425 w_4 + 2.8432 w_5$

$\geq 0.25 w_1 + 0.16 w_2 + 0.27 w_3 + 0.15 w_4 + 0.2 w_5$

$s:t \, w_4 + w_5 + w_3 \geq 18; \, w_5 \geq 35; \, w_2 \geq w_5 \geq w_3 \geq (4.5)$

$> n$

$> n$
\[ w_j = 1 \]
\[ \sum_{j=1}^{n} (w_j)^T = (.25, .16, .165, .1, .325)^T \]

Step 5: Calculation of closeness degree of the alternatives.
\[ k(a_1) = .6521, \quad k(a_2) = .0799, \quad k(a_3) = .8928, \quad k(a_4) = .5322. \]

Step 6: Rank the alternatives.
\[ a_3 > a_1 > a_4 > a_2. \] Hence, optimum alternative is \( a_3 \).

The ranking of four alternatives is also acquired by the TOPSIS, F-TOPSIS, intuitionistic fuzzy TOPSIS and proposed, methods and is compared in Table 6 given below:

<table>
<thead>
<tr>
<th>Methods</th>
<th>Ranking</th>
<th>Optimal alternative</th>
</tr>
</thead>
<tbody>
<tr>
<td>TOPSIS method proposed by Hwang, Yoon (1981)</td>
<td>( a_3 &gt; a_4 &gt; a_1 &gt; a_2 )</td>
<td>( a_3 )</td>
</tr>
<tr>
<td>F-TOPSIS proposed by Chen (2001)</td>
<td>( a_3 &gt; a_1 &gt; a_4 &gt; a_2 )</td>
<td>( a_3 )</td>
</tr>
<tr>
<td>IF-TOPSIS proposed by Joshi, Kumar (2014)</td>
<td>( a_3 &gt; a_1 &gt; a_4 &gt; a_2 )</td>
<td>( a_3 )</td>
</tr>
<tr>
<td>Mishra et al method (2017)</td>
<td>( a_3 &gt; a_1 &gt; a_4 &gt; a_2 )</td>
<td>( a_3 )</td>
</tr>
<tr>
<td>D.S. Hooda method (2018)</td>
<td>( a_3 &gt; a_1 &gt; a_4 &gt; a_2 )</td>
<td>( a_3 )</td>
</tr>
<tr>
<td>Proposed method</td>
<td>( a_3 &gt; a_1 &gt; a_4 &gt; a_2 )</td>
<td>( a_3 )</td>
</tr>
</tbody>
</table>

It is worth mentioning that there is no discrepancy in the ranking order of the alternatives by TOPSIS method, F-TOPSIS method, IF-TOPSIS method, Mishra et al (2017), Hooda (2018) and proposed method.

4.2 Pattern Recognition

Now, we will discuss how pattern recognition become simple by demonstrating application of the proposed generalised fuzzy divergence measure given as below.

Assume, we are have \( m \) known patterns \( P_1, P_2, P_3, ..., P_m \) having the classes \( D_1, D_2, D_3, ..., D_m \), respectively.

The Pattern are represented by the following fuzzy set in the universe of discourse \( K = f_kl, k_2, k_3, ..., k_og \):
\[ P_i = f h k_i, \quad p_i(k_j) i / k_j 2 K g \]

where \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n \).

Given an unknown pattern \( Q \), represented by the fuzzy set \( Q_i = f h k_i, q_i(k_j)i / k_j 2 K g \)

Our main motive is to classify \( Q \) to one of the classes \( D_1, D_2, D_3, \ldots, D_m \). The process of assigning \( Q \) to \( D_X \) described below is as per the principle of minimum divergence/discrimination information between fuzzy sets

\[ X = \text{arg min} fM(P_X, Q)g. \]

Above algorithm, the given pattern can be recognized so that the best class can be selected. It is considered as a practical application of minimum divergence measure principal [22] for pattern recognition.

Case Study Consider a problem having four known patterns \( P_1, P_2, P_3 \) and \( P_4 \) which have classifications \( D_1, D_2, D_3 \) and \( D_4 \) respectively. These are represented by the following fuzzy sets in the universe of discourse \( K = f k_1, k_2, k_3g. \)

\[ P_1 = f h k_1, .7i, h k_2, .3i, h k_3, .1ig \]
\[ P_2 = f h k_1, .4i, h k_2, .2i, h k_3, .5ig \]
\[ P_3 = f h k_1, .6i, h k_2, .3i, h k_3, .8ig \]
\[ P_4 = f h k_1, .7i, h k_2, .5i, h k_3, .8ig \]

Given an unknown pattern \( Q \), represented by the fuzzy set

\[ Q = f h k_1, .5i, h k_2, .4i, h k_3, .9ig \]

Table 7: Calculated numerical values of \( N_t(P_X, Q) \), \( X = 1, 2, 3, 4 \) for any \( t 0 \)

<table>
<thead>
<tr>
<th>( i )</th>
<th>1</th>
<th>2</th>
<th>10</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_t(P_1, Q) )</td>
<td>1.83275</td>
<td>3.16293</td>
<td>315.33686</td>
<td>3117982410207.97</td>
</tr>
<tr>
<td>( N_t(P_2, Q) )</td>
<td>36451</td>
<td>20263</td>
<td>.03221</td>
<td>4.28174e-06</td>
</tr>
<tr>
<td>( N_t(P_3, Q) )</td>
<td>0.4307</td>
<td>0.0258</td>
<td>1.39698e-10</td>
<td>1.05097e-46</td>
</tr>
<tr>
<td>( N_t(P_4, Q) )</td>
<td>0.07572</td>
<td>0.00474</td>
<td>1.39698e-10</td>
<td>1.05097e-46</td>
</tr>
</tbody>
</table>

Our aim here is to classify \( Q \) to one of the classes \( D_1, D_2, D_3 \) and \( D_4 \). From the formula (2.1), we can compute the values of generalized fuzzy divergence measure \( N_t(P_i, Q) \), \( i = f 1, 2, 3, 4g \) for any \( t 0 \) and are presented in Table (4) as follows. It is observed that \( Q \) can be classified to \( D_3 \) correctly.

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4.3 Medical Diagnosis

The application of fuzzy sets theory has been found in the diagnosis of disease by many researchers. Pattern recognition algorithms is appropriate tool to re-solve medical diagnosis problems for recognizing disease which is a challenging research area according to practical point of view.

Below example elaborate the e ciency of the proposed generalized fuzzy di-vergence measure in re-solving medical diagnosis problems, using the algorithm given by Ohlan [17].

Suppose that the universe of discourse X is a set of symptoms

\[ \mathbf{K} = k_1 \text{(Temperature)}, k_2 \text{(Headache)}, k_3 \text{(Stomach Pain)}, k_4 \text{(Cough)}, k_5 \text{(Chest Pain)} \]

Consider a set of diagnosis

\[ \mathbf{B} = B_1 \text{(Viral)}, B_2 \text{(Malaria)}, B_3 \text{(Typhoid)}, B_4 \text{(Stomach Problem)}, B_5 \text{(Chest Problem)} \]

whose elements are presented by the following FSs, respectively,

\[ B_1 = f<k_1, 0.7>, <k_2, 0.2>, <k_3, 0.0>, <k_4, 0.7>, <k_5, 0.1> \]
\[ B_2 = f<k_1, 0.4>, <k_2, 0.3>, <k_3, 0.1>, <k_4, 0.4>, <k_5, 0.1> \]
\[ B_3 = f<k_1, 0.1>, <k_2, 0.2>, <k_3, 0.8>, <k_4, 0.2>, <k_5, 0.2> \]
\[ B_4 = f<k_1, 0.1>, <k_2, 0.0>, <k_3, 0.2>, <k_4, 0.2>, <k_5, 0.8> \]
\[ B_5 = f<k_1, 0.3>, <k_2, 0.6>, <k_3, 0.2>, <k_4, 0.2>, <k_5, 0.1> \]

Table 8: Calculated numerical values of \( N_t(P, B_i) \), \( i = f1, 2, 3, 4g \) for any \( t \)

<table>
<thead>
<tr>
<th></th>
<th>( t = 2 )</th>
<th>( t = 3 )</th>
<th>( t = 10 )</th>
<th>( t = 50 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_t(P, B_1) )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( N_t(P, B_2) )</td>
<td>0.45538</td>
<td>0.41498</td>
<td>0.52645</td>
<td>2.69467</td>
</tr>
<tr>
<td>( N_t(P, B_3) )</td>
<td>1.37102</td>
<td>1.80589</td>
<td>31.90312</td>
<td>804484348.54885</td>
</tr>
<tr>
<td>( N_t(P, B_4) )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( N_t(P, B_5) )</td>
<td>2.8844</td>
<td>0.09772</td>
<td>0.0014</td>
<td>1.38297e-18</td>
</tr>
</tbody>
</table>

The aim here is to assign a patient

\[ P = f<k_1, .8>, <k_3, .2>, <k_3, .6>, <k_4, .6>, <k_5, .1> \]

, to one of the above mentioned diagnosis \( B_1, B_2, B_3, B_4 \) and \( B_5 \). We proceed by considering the criteria \( \min N_t(P, B_i) \) with \( t \geq N \)

\[ i \]

Table (5) presents the values of using \( N_t(P, B_i) \), \( i = f1, 2, 3, 4g \) for \( t \geq N \) using the measure (2.1). It has been observed that the proper diagnosis for patient \( P \) is \( B_5 \) (Chest problem).
Results and Discussion

In some engineering problems, the decision based on crisp values is not possible. Therefore, the use of fuzzy MCDM may be more beneficial. The proposed approach can be used in various decision-making problems, PR and MD. The most of the existing MCDM approaches do not consider the vagueness and uncertainty in the problem and fail to make a good decision. Here MCDM, PR and MD will consider both the gaps, it can be further used in other decision-making problems like supplier selection, project selection, location, face detection, medical and other such problems.

Conclusion

Same techniques may be adopt for other decision-making, medical problems like supplier selection, project selection, location selection and others more real-life problems. Proposed symmetric fuzzy divergence measure with proof of validity and some more efficient properties of this divergence measure also proven. Thus, it is concluded that the proposed divergence measure and method of MCDM, PR and MD require no computation. Finally, we observe that the proposed divergence measure is very appropriate measure to solve real-world problems related to MCDM, PR and MD.

REFERENCES:

19. Saraswat R.N. and N. Khatod, Symmetric Fuzzy Divergence Measure, Decision Making and Medical Diagnosis Problems, research paper accepted in Journal of Intelligent and Fuzzy System 2019

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